Lecture # 33

Application to the Definite Integral

Application of the Definite Integral

- Area problem: Area Between two Curves
- Area between y = f(x) and y = g(x)
- Area between x = v(y) and x = w(y)

First Area problem

Area between two curves:

Area between y=f(x) and y=g(x)

Suppose that *f* and *g* are continuous functions on an interval [a, b] and

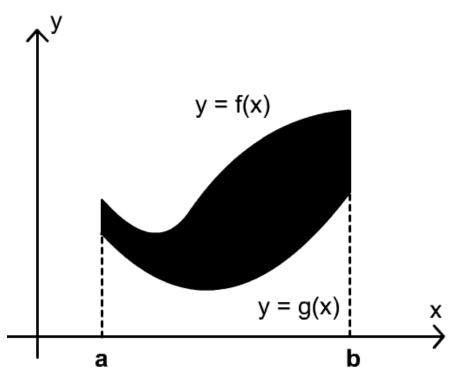
$f(x) \geq g(x)$ for $a \leq x \leq b$

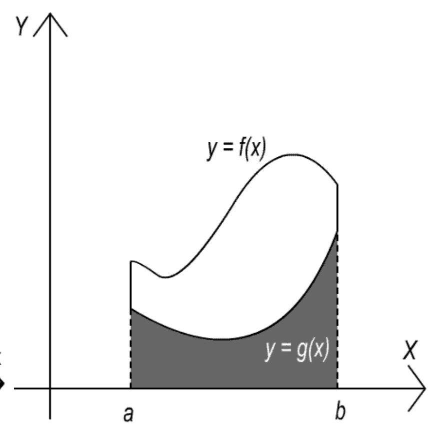
(This means that the curve y = f (x) is above y = g (x) and that the two can touch but not cross).

Find the area A of the region bounded above by y= f (x), below by y = g(x), and on the sides by the lines x = a and x = b

If f and g are nonnegative on [a,b] then we have

A = [area under f] – [area under g]





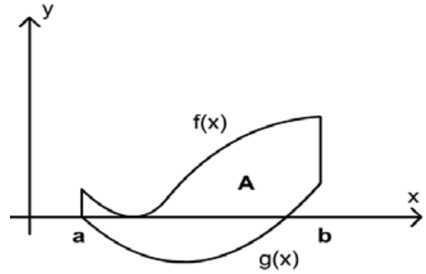
In terms of integrals we can say this as $A = \int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b [f(x) - g(x)]dx$

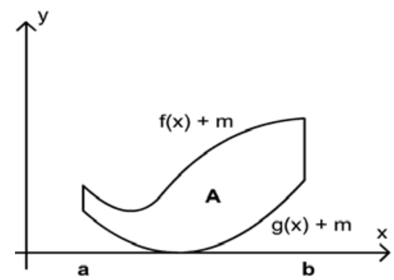
What if f and g take on negative values too?

This means that their graphs are below the x-axis for some values in [a,b].

This can be remedied if we translate the two graphs by a constant so big that it shifts both f and g above the x-axis.

This shift does not affect the area between the two curves





$$A=\int_a^b[f(x)+m]dx-\int_a^b[g(x)+m]dx=\int_a^b[f(x)-g(x)]dx$$

The constants cancel each other in the calculations.

Here is the formal definition

DEFINITION 6.1. (Area Formula)

If f and g are continuous functions on the interval [a, b], and $f(x)\geq g(x)$ for all x in [a, b], then the area of the region bounded above by $y=f(x)$, below by $y=g(x)$, on the left by the line x=a, and on the right of the line x=b is

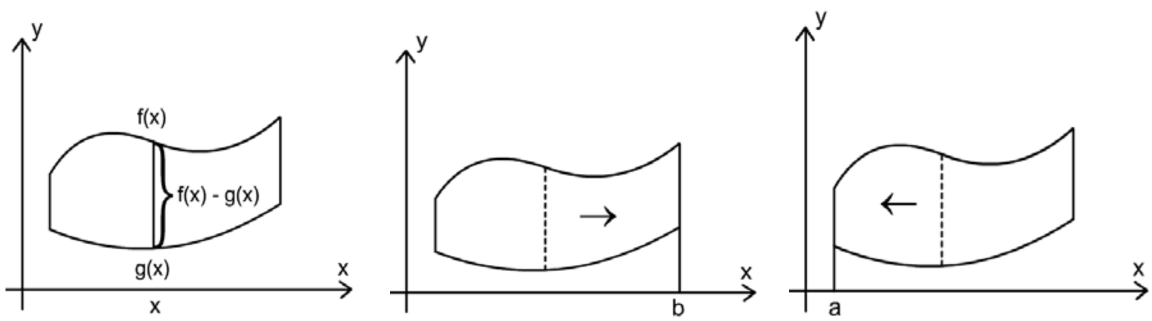
$$A=\int_a^b[f(x)-g(x)]dx$$

A few things to keep in mind.

If the region confined by two curves is complicated, then it may require some careful thought to determine the integrand.

It may be hard to find the limits of integration.

Here is a procedure we can follow



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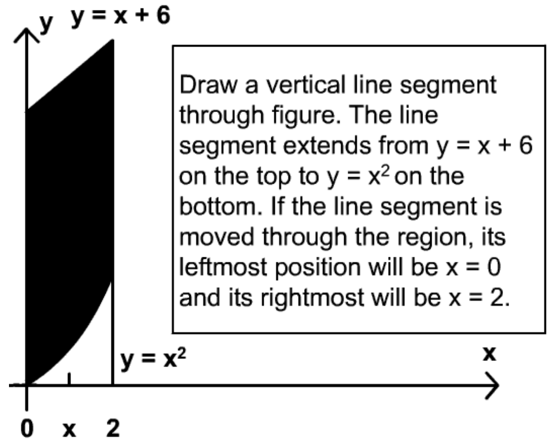
Example

Find the area of the region bounded above by y = x + 6, and below by

$y = x^2$ and on the sides by lines x = 0 and x = 2.

Solution:

The region and a vertical line through it are shown in this figure.



The line extends from f(x) = x + 6 on the top to $y = x^2$ at the bottom.

As the line moves left to right, its leftmost position will be x = 0 and right most will be x = 2

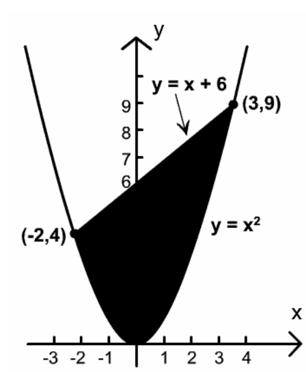
So we have

$$A = \int_0^2 [(x+6) - x^2]dx = \left.\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_0^2 = \frac{34}{3} - 0 = \frac{34}{3}$$

Example

Find the area of the region that is enclosed between $y = x^2$ and y = x + 6.

solution:



There are no lines at the right or the left. So how do we find the limits of integration? Well here is a sketch of the region.

Note that the integration limits are defined by the points where the two curves intersect.

We can find these points by equating the two equations and solving for x

We get

$x^2 = x + 6$

$\Rightarrow \; x^2 - x - 6 = 0$

$\Rightarrow \; (x+2)(x-3) = 0$

$\Rightarrow \; x = -2 \quad \text{and} \quad x = 3$

So now we have

$$A = \int_{-2}^{3} [(x+6) - x^2]dx = \frac{x^2}{2} + 6x - \frac{x^3}{3}\Big]_{-2}^{3} = \frac{125}{6}$$

Example

Find the area of the region enclosed by $x = y^2$ and y = x –2

Solution:

First we need to know where the two curves intersect.

Let's write down y = x – 2 as x = y + 2 and then equate

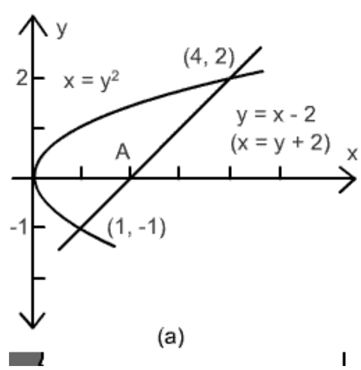
$$x = y^2 \text{ and } x = y + 2$$

This gives

$$y^2 = y + 2 \;\Rightarrow\; y^2 - y - 2 = 0 \;\Rightarrow\; y = -1 \text{ and } y = 2$$

Substituting these into the original equations will give us the desired x-values, which are x = 1 and x = 4 respectively

Here is the graph of the situation



From this figure, we see that the upper boundary or curve is defined by the equation y = +√(x).

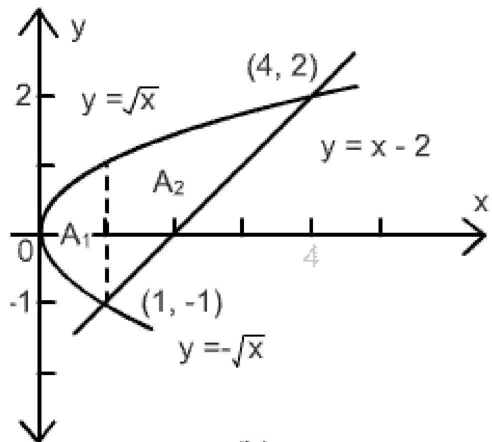
The lower boundary has two pieces

1) y= -√(x) for x between –1 and 0
2) y= x-2 for x between1 and 4

This causes us problems when we try to determine f(x) – g(x) for the integration.

Instead we will divide the region into two parts and evaluate the area of each region and then add together.

Here is how we will divide it



(b)

f(x) = +√(x), g(x) = -√(x), a = 0, b = 1

$$A_1 = \int_0^1 \left[\sqrt{x} - (-\sqrt{x})\right] dx = 2\int_0^1 \sqrt{x}\,dx$$

$$= 2\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^1 = \frac{4}{3}$$

f(x) = +√ (x), g(x) = x – 2, a = 1, b = 4

$$A_2 = \int_1^4 \left[\sqrt{x} - (x-2)\right] dx = \int_1^4 (\sqrt{x} - x + 2)\,dx$$

$$= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{2}x^2 + 2x\right]_1^4 = \frac{19}{6}$$

Total area or region is 19/6 + 4/3 = 9/2

Area between x = v(y) and x = w(y)

We can avoid splitting the region if we integrate w.r.t y instead of x.

Second Area problem

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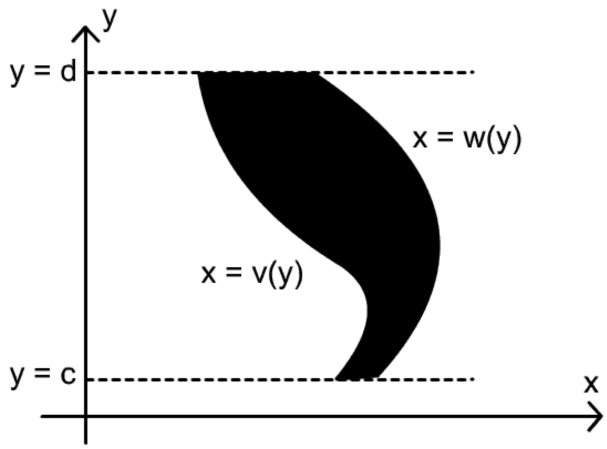


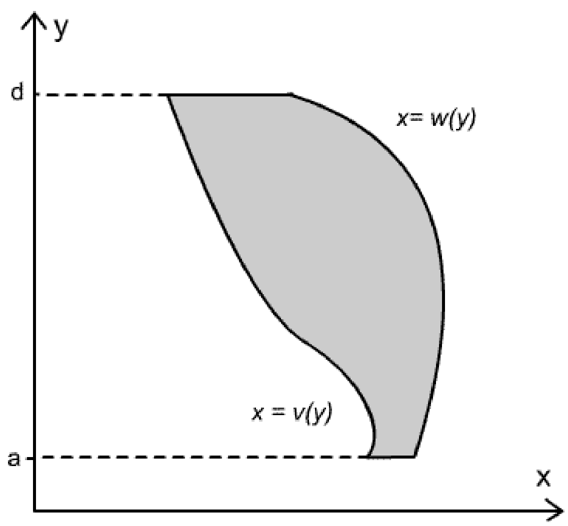
Just as we derived an answer earlier in the lecture for the first area problem for regions confined by two curves, we get the following definition

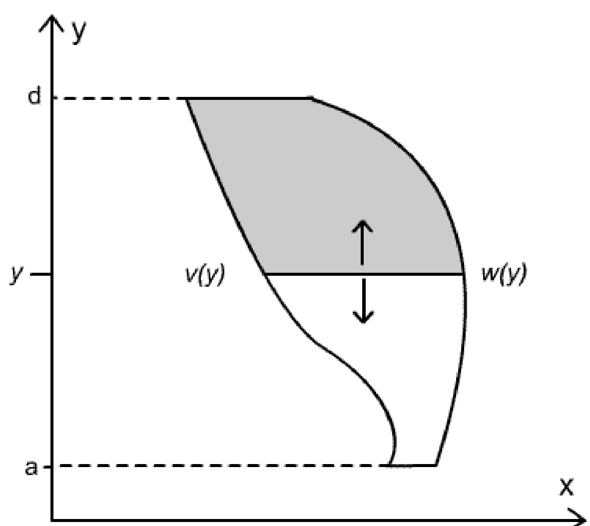
DEFINITION 6.1.2 (Area Formula)

If w and v are continuous functions and if $w(y) \geq v(y)$ for all y in [c, d], then the area of the region bounded on the left by $x = v(y)$, on the right by $x = w(y)$, below by y = c, and above by y = d is

$$A = \int_c^d [w(x) - v(x)]dy$$

The procedure for finding the integrand and limits of integration is the same as earlier. Here are two figures to help you see this





Example

Find the area of the region confined by $x = y^2$ and y=x-2 or x=y+2

Solution:

Left boundary is $x = y^2$

Right boundary is x = y + 2

Limits are from –1 to 2

$$A = \int_{-1}^{2} [(y+2) - y^2]dy = \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_{-1}^{3} = \frac{9}{2}$$

The area is shown in the figure ab