Lecture # 32
Second Fundamental Theorem of Calculus

Second Fundamental Theorem of Calculus

- Dummy Variable
- Definite Integrals with variable upper limit
- Second fundamental theorem of Calculus
- Existence of Ant derivatives for continuous functions
- Functions defined by Integrals

Dummy Variable

Here is some notational stuff.

If we change the letter for the variable of integration but don't change the limits, then the values of the definite integral are unchanged.

That is to say

$$\int_a^b f(x)dx, \quad \int_a^b f(t)dt, \quad \int_a^b f(y)dy$$

All have the same value

For this reason, we call the letter used for the variable of integration DUMMY variable

Example

$$\int_1^3 x^2 dx = \left.\frac{x^3}{3}\right]_1^2 = \frac{26}{3}$$

$$\int_1^3 f(t)dt = \left.\frac{t^3}{3}\right]_1^3 = \frac{26}{3}$$

$$\int_1^3 f(y)dy = \left.\frac{y^3}{3}\right]_1^3 = \frac{26}{3}$$

Definite Integrals with variable upper limit of integration

We will now consider definite integrals of the form $\int_a^x --$ where the upper limit is a variable rather than a number.

In such integrals, we will use a different letter for the integration variable

This will distinguish between the limit of integration and the variable of integration.

Let's do an example

Example

Evaluate

$$\int_2^x t^2 dt$$

Solution:

$$\int_2^x t^2 dt = \left.\frac{t^3}{3}\right]_2^x = \frac{x^3}{3} - \frac{8}{3}$$

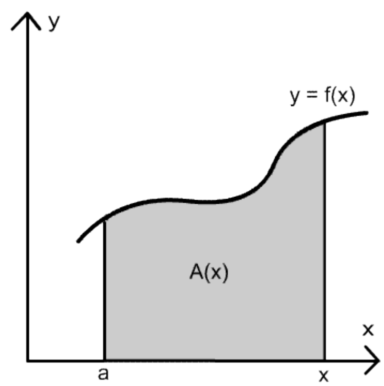
Note that the result in this case is a FUNCTION of x.

In all such integrals with an x as a limit, the final result is a function of x.

This is like when we said that instead of evaluating the derivative at a number, do it at a point x to get the derivative function.

Second fundamental theorem of Calculus

In lecture 25, we say that if f is a nonnegative continuous function over [a, b], and A (x) represents the area under the curve y = f (x) over the interval [a ,x] as a function of x, then $A'(x) = f(x)$



Now Let's write A (x) as a definite integral since that's how it is represented

$$A(x) = \int_a^x f(t)dt$$

If we now take the derivative w.r.t to x , we get

$$\frac{d}{dx}[A(x)] = A'(x) = \frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$$

This result holds for all continuous functions, and we can state it as a theorem

Theorem 5.9.1 (2nd fundamental theorem of Calculus)

Read as

1) If the integrand is continuous, then the derivative of a definite integral w.r.t its upper limit is equal to the integrand evaluated at the upper limit.
2) The derivative of a function representing the area under the curve of another continuous function f is equal to the function f on a given interval.

We usually write the theorem as

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$$

Example

Since $f(x) = x^3$ is continuous everywhere, evaluate the integral $\int_1^x t^3 dt$ to check the validity of 2nd fundamental theorem of calculus.

Solution:

By 2nd fundamental theorem of calculus:

$$\frac{d}{dx}\left[\int_1^x t^3 dt\right] = x^3$$

We check it by evaluating the integral

$$\int_1^x t^3 dt = \left.\frac{t^4}{4}\right]_1^x = \frac{x^4}{4} - \frac{1}{4}$$

Differentiating this gives x^3.

Existence of Anti-derivatives of continuous functions:

If 'f is a continuous function on an interval 'I' and 'a' is any point in 'I' then this formula

$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$ tells us that $F(x) = \int_a^x f(t)dt$ is an anti-derivative of f on I. So now with the 2nd theorem of calculus, we can say that EVERY continuous function on an interval has an anti-derivative on that interval.

Functions defined by Integrals

So far we have been able to determine the anti-derivative of a given function by integration techniques.

But its not always possible to find the anti-derivative of a given function *f* which is continuous on an interval and thus is guaranteed to have an anti-derivative on that interval.

If this is the case, we get functions (anti-derivatives) that are defined in terms of integrals and nothing simpler that we have seen so far.

Example

$$\int_1^3 \frac{1}{x}dx$$

Solution:

This function is continuous on the interval [1, 3] so it is integrable on [1, 3]. By the 2nd theorem of calculus, an antiderivative is

$$F(x) = \int_1^x \frac{1}{t}dt$$

Using the 1st theorem of calculus we can attempt to evaluate this integral

$$\int_1^x \frac{1}{t}dt = F(3) - F(1) = \int_1^3 \frac{1}{t}dt - \int_1^1 \frac{1}{t}dt = \int_1^3 \frac{1}{t}dt$$

But this is not a function like any we have seen so far.

In fact, 1/t cannot be integrated using polynomials, rational functions, or any other function we have seen so far.

But we can approximate this function by using numerical methods like the Riemann sums. Now this formula $\int_1^x \frac{1}{t}dt$ defines a FUNCTION in terms of integrals.