Lecture # 31

Evaluating Definite Integral by Substitution.

Evaluating Definite Integrals.

- By substitution
- Approximation by Riemann Sums

Evaluating definite integral by substitution

We want to evaluate the following definite integral by substitution

$$\int_a^b f(x)dx$$

Method 1

Evaluate the indefinite integral $\int f(x)dx$ by substitution , and then use the relationship.

$$\int_a^b f(x)dx = \left[\int f(x)dx\right]_a^b$$

Method 2

First represent the definite integral in the form $\int_a^b f(x)dx = \int_a^b h(g(x))g'(x)dx$

And then make the substitution $u = g(x), \quad du = g'(x)dx$ directly into the definite integral. But now we have to change integration limits from x-limits to u-limits as follows

$$u = g(a), \quad \text{if } x = a$$
$$u = g(b), \quad \text{if } x = b$$

This leaves us with a new integral in terms of u

$$\int_a^b f(x)dx = \int_{g(a)}^{g(b)} h(u)du$$

This will be simpler if the choice of u is good.

Example

Evaluate $\int_0^2 x(x^2+1)^3 dx$

Solution:

Method 1

$u = x^2 + 1$ so $du = 2xdx$

$$\int x(x^2+1)^3 dx = \frac{1}{2}\int u^3 du = \frac{u^4}{8} + C = \frac{(x^2+1)^4}{8} + C$$

So

$$\int_0^2 x(x^2+1)^3 dx = \left[\int x(x^2+1)^3 dx\right]_0^2 = \left.\frac{(x^2+1)^4}{8}\right]_0^2 = 78$$

Method 2

$u = 1$ if $x = 0$

$u = 5$ if $x = 2$

Thus

$$\int_0^2 x(x^2+1)^3 dx = \frac{1}{2}\int_1^5 u^3 du = \left.\frac{u^4}{8}\right]_1^5 = 78$$

Example

$$\int_0^{\frac{\pi}{4}} \cos(\pi - x)dx$$

Solution:

Let $u = \pi - x$ so that $du = -dx$

Also $u = \pi$ if $x = 0$, $u = \frac{3\pi}{4}$ if $x = \frac{\pi}{4}$

So,

$$\int_0^{\frac{\pi}{4}} \cos(\pi - x)dx = -\int_\pi^{\frac{3\pi}{4}} \cos(u)(du) = -\sin(u)\Big]_\pi^{\frac{3\pi}{4}} = -[\sin(\frac{3\pi}{4}) - \sin(\pi)]$$

$$= -[\frac{1}{\sqrt{2}} - 0] = -\frac{1}{\sqrt{2}}$$

Approximation by Riemann Sums

Recall that a Riemann Sum is the expression $\sum_{k=1}^{n} f(x_k^*)\Delta x_k$

which occurs when we try to approximate the area under the curve.

If we take the limits of this sum, we get the definite integral.

If we don't take the limits, then we can get a good approximation to the definite integral if n is relatively LARGE.

So we can say that for large n $\int_a^b f(x)dx \approx \sum_{k=1}^{n} f(x_k^*)\Delta x_k$

We need to use this approx when it's impossible to evaluate exactly.

It is a good idea in this case to use a regular partition of the interval [a,b] which gives same width for each subinterval.

So we then write

$$\int_a^b f(x)dx \approx \sum_{k=1}^{n} f(x_k^*)\Delta x_k = \Delta x_k \left[f(x_1^*) + f(x_2^*) + \ldots f(x_n^*) \right]$$

This formula produces a left endpoint approximation, a right end point approximation and midpoint approximation , depending on the choice of the point x_k^*

Example

Approximate $\int_0^1 \sqrt{1-x^2}\,dx$ using the left endpoint, right end point, and the midpoint approx each with n=10, n=20, n=50 and n=100 subintervals.

Solution:

For n =10 we can do this easily, but for others we should use a computer. We will do for n=10 using the ideas from previous lectures. Here is a table of results

n	Left endpoint approximation	Right endpoint approximation	Midpoint approximation
10	0.826129582	0.726129582	0.788102858
20	0.807116220	0.757116220	0.786357647
50	0.794567128	0.774567128	0.785641388
100	0.790104258	0.780104258	0.785484214

The exact value of the integral is $\frac{\pi}{4}$. This is consistent with the preceding computations , since

$\frac{\pi}{4} \approx 0.785398163$.