Lecture # 30

First Fundamental Theorem of Calculus

- 1st fundamental theorem of calculus.
- Relationship between definite and indefinite integrals.
- Mean Values theorem for Integrals.
- Average Values of a function.

This is the 1st fundamental theorem of calculus.

Theorem 5.7.1 (The first fundamental theorem of calculus)

If f is continuous on $[a,b]$ and if F is an ant-derivative of f on $[a,b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

This theorem tells us how to evaluate EASILY the definite integral.
It says that find the anti-derivative of $f(x)$ first, call it $F(x)$, and then evaluate this function on the limits of the integration. Let's prove this.

Proof
We will use the Mean Value Theorem involving derivatives to prove the first fundamental theorem of calculus.

$$\int_a^b f(x)dx = F(b) - F(a)$$

Let us subdivide the given interval of integration namely $[a,b]$ into n subintervals using the points $x_1, x_2, ..., x_{n-1}$ in the interval $[a,b]$ such that
$a < x_1 < x_2 < ... < x_{n-1} < b$.
So now we have n subinterval of $[a,b]$. We can denote the widths of these intervals as $\Delta x_1, \Delta x_2, ..., \Delta x_n$. Where for example $\Delta x_2 = x_2 - x_1$.
Since $F'(x) = f(x)$ for all x in $[a,b]$, it is obvious that $F(x)$ satisfies the requirements of the Mean Value Theorem involving derivatives on each of the subintervals. So by MVT, we can find points $x_1^*, x_2^*, ..., x_n^*$
In each of the respective subintervals $[a, x_1], [x_1, x_2], ..., [x_{n-1}, b]$.

So we now make the following equations

$$F(x_1)-F(a)=F'(x_1^*)(x_1-a)=f(x_1^*)\Delta x_1$$
$$F(x_2)-F(x_1)=F'(x_2^*)(x_2-x_1)=f(x_2^*)\Delta x_2$$
$$F(x_3)-F(x_2)=F'(x_3^*)(x_3-x_2)=f(x_3^*)\Delta x_3$$
.
.
.
$$F(b)-F(x_{n-1})=F'(x_n^*)(b-x_{n-1})=f(x_n^*)\Delta x_n$$

Adding up these equations we get

$$F(b)-F(a)=\sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

Now increase n in such a way that

$$\max \Delta x_k \to 0$$

Since f is assumed to be continuous, we have the following result

$$F(b)-F(a)=\lim_{\max \Delta x_k \to 0}\sum_{k=1}^{n} f(x_k^*)\Delta x_k = \underset{\substack{\uparrow \\ \text{THEOREM 5.6.9(a)}}}{\int_a^b f(x)dx}$$

I did not apply the limits on the left because even if I did, the expression does not involve *n* and therefore nothing happens.

$F(b)-F(a)$ can also be written as $F(x)]_a^b$ so we have

$$\int_a^b f(x)dx = F(x)]_a^b$$

Example

Evaluate $\int_1^2 xdx$

$\int_1^2 xdx$ .The function $F(x)=\frac{1}{2}x^2$ is and antiderivative of $f(x)=x$ .So we have

$$\int_1^2 xdx = \frac{1}{2}x^2\Big]_1^2 = \frac{1}{2}(4)-\frac{1}{2}(1)=2-\frac{1}{2}=\frac{3}{2}$$

Here are a few properties of the bracket notation we just saw.
Prove these yourself

Properties

$$\left[cF(x)\right]_a^b = c\left[F(x)\right]_a^b$$
$$\left[F(x)+G(x)\right]_a^b = \left[F(x)\right]_a^b + \left[G(x)\right]_a^b$$
$$\left[F(x)-G(x)\right]_a^b = \left[F(x)\right]_a^b - \left[G(x)\right]_a^b$$

These are easy if you remember that this notation and the definite integral are the same thing!!
Relationship between definite and indefinite integrals
In applying the 1st theorem of Calc, it does not matter WHICH anti-derivative of f is used.
If F is any anti-derivative of f then all the others have form
F(x) + C by theorem 5.2.2
Hence we have the following:

$$\left[F(x)+C\right]_a^b = \left[F(b)+C\right]-\left[F(a)+C\right]$$

$$= F(b)-F(a) = F(x)\Big]_a^b = \int_a^b f(x)dx \qquad \ldots\text{(A)}$$

This shows that all anti-derivatives of f on [a,b] give the same values for the definite integral
Now since

$$\int f(x)dx = F(x)+C$$

It follows from (A) that

$$\int_a^b f(x)dx = \left[\int f(x)dx\right]_a^b$$

So we can first evaluate the indefinite integral and use the limits as well to evaluate the definite integral
This relates the definite and the indefinite integrals.

Example
Using the 1st theorem of calculus, find the area under the curve
y = cos(x) over the interval [0, 2π]
Since $\cos(x)\geq 0$ for $0 \leq x \leq \pi$, the area is

$$A = \int_0^{\frac{\pi}{2}} \cos(x)dx = \left[\int \cos(x)dx\right]_0^{\frac{\pi}{2}}$$

$$= \sin(x)\Big]_0^{\frac{\pi}{2}} = \sin(\frac{\pi}{2}) - \sin(0) = 1$$

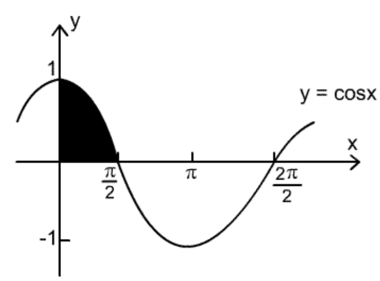


Figure shows that cosx ≥ 0 for 0 ≤ x ≤ π/2

I deliberately chose C = 0 since we just saw that the value of C=0 doesn't change the answer.

Example

Evaluate $\int_0^3 (x^3 - 4x + 1)dx$

$$\int_0^3 (x^3 - 4x + 1)dx = \left[\int (x^3 - 4x + 1)dx \right]_0^3$$
$$= \left[\int (x^3 dx - \int 4x.dx + \int 1.dx \right]_0^3$$
$$= \left[\frac{x^4}{4} - 4.\frac{x^2}{2} + x \right]_0^3 = (\frac{81}{4} - 18 + 3) - (0) = \frac{21}{4}$$

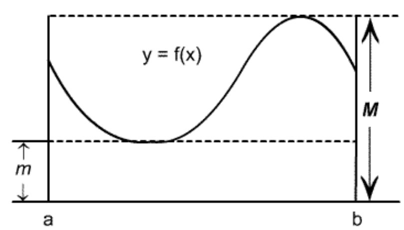
Example

Evaluate $\int_0^6 f(x)dx$ if $f(x) = \begin{cases} x^2 & x < 2 \\ 3x - 2 & x \geq 2 \end{cases}$

From theorem 5.6.6

$$\int_0^6 f(x)dx \text{ if } f(x) = \int_0^2 f(x)dx + \int_2^6 f(x)dx$$
$$= \int_0^2 x^2 dx + \int_2^6 (3x - 2)dx$$
$$= \left.\frac{x^3}{3}\right]_0^2 + \left[\frac{3x^2}{2} - 2x \right]_2^6 = \left(\frac{8}{3} - 0 \right) + (42 - 2) = \frac{128}{3}$$

Mean Values theorem for Integrals



Consider the picture.
This figure shows a continuous function *f* on [a,b].
Let M be the maximum values of *f* on [a,b], and m the minimum.
Then if we draw a rectangle of height M and one of height m, then its clear from the picture that the area under the curve of *f* is at least as large as the area of rectangle with height m AND no larger than the area of the rectangle with height M.
So we would like to say that there is some rectangle of a certain height for which the area under *f* = area of rectangle.

THEOREM 5.7.2(The Mean-Value Theorem for Integrals)

If f is continuous on a closed interval [a, b], then there is at least one number x^* in [a, b]such that

$$\int_a^b f(x)dx = f(x^*)(b - a)$$

Proof

By the extreme value theorem (theorem 4.6.4) *f* assumes a max M and a min m on [a,b]. So for all x in [a,b]

$$m \le f(x) \le M$$

$$\Rightarrow \int_a^b m\,dx \le \int_a^b f(x)dx \le \int_a^b M\,dx$$ Theorem 5.6.7(b)

$$\Rightarrow m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$

$$\Rightarrow m \le \frac{1}{b-a}\int_a^b f(x)dx \le M$$ Evaluating the end integral

The last inequality states that $\frac{1}{b-a}\int_a^b f(x)dx$ is a number btw m and M on [a,b]

Since *f* is continuous and takes on all values in [a,b], we can say using the Intermediate Values theorem (theorem 2.7.9) that *f* must assume $\frac{1}{b-a}\int_a^b f(x)dx$ on [a,b] for some point x*

So we have

$$\frac{1}{b-a}\int_a^b f(x)dx = f(x^*)$$

$$\Rightarrow \int_a^b f(x)dx = f(x^*)(b-a)$$

Example

$f(x) = x^2$ is continuous on [1, 4], the MVT for Integrals guarantees that there exists a number x^* in [1, 4] such that

$$\int_1^4 x^2 dx = f(x^*)(4-1) = 3(x^*)^2$$

But

$$\int_1^4 x^2 dx = \left.\frac{x^3}{4}\right]_1^4 = 21$$

So

$$3(x^*)^2 = 21 \Rightarrow x^* = \sqrt{7}$$

Average Values of of a function

Definition 5.7.3

If f is integrable on [a, b] then the average value (or mean value) of f on [a, b] is defined to be $f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx$

If $y = f(x)$, then f_{avg} is also called the average value of y with respect to x over [a, b].