Lecture # 29

The Definite Integral

- Definition of Definite Integral
- Definite Integral of continuous functions with nonnegative values
- Definite Integral of continuous functions with negative and positive values
- Definite Integral of functions with discontinuities
- Properties of the Definite Integral
- Inequalities involving Definite Integral

We have so far developed a definition of Area as limit of a bunch of rectangles with equal widths. The question arises that why choose widths? The answer is that we don't have to.

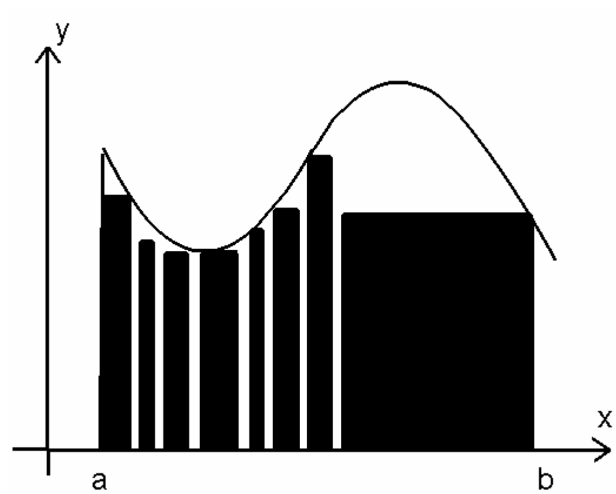
We want now to have a definition of Area that includes the general case where the widths of the rectangles are not necessarily equal.

When the rectangles widths were equal note that the formula for the width was defined as

$$\Delta x = \frac{b-a}{n}$$

In this formula you can see that as *n* goes to infinity, the width delta x goes to zero

If the widths are not the same for all rectangles, then this is not necessarily the case. Suppose that we have a rectangle construction in which we divide [a,b] so that one half is continually subdivided into smaller Rectangle, while the right is left as one big rectangle.

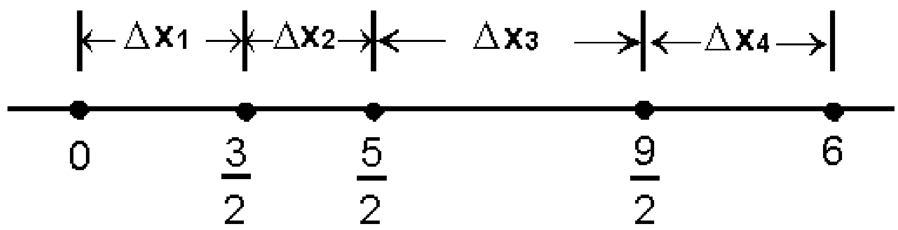


In this case, when n goes to inf, left part goes to 0, not the Right half.

Let's fix this problem.

Subdivide [a,b] into *n* subinterval whose widths are $\Delta x_1, \Delta x_2, \Delta x_3, ..., \Delta x_n$

The subintervals are said to partition the interval, and the largest subinterval width is called the mesh size of the partition. This is denoted by $\max \Delta x_k$



$$\max \Delta x_k = \Delta x_3 = \frac{9}{2} - \frac{5}{2} = 2$$

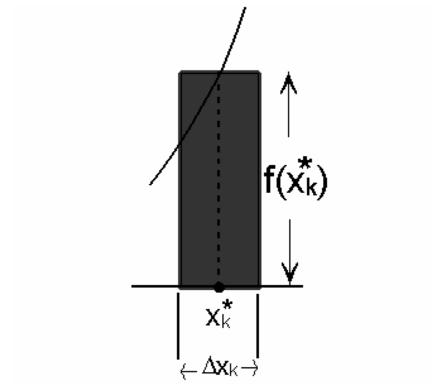
Read as "maximum of the Δx_k."

In this figure

$$\max \Delta x_k = \Delta x_3 = \frac{9}{2} - \frac{5}{2} = 2$$

We see that

If [a,b] is partitioned into n subintervals, and if x_k is an arbitrary point in kth subinterval, then $f(x_k^*)\Delta x_k$ is the area of the rectangle with height f(x_k) and width delta x_k, and $\sum_{k=1}^{n} f(x_k^*)\Delta x_k$ is the sum of the shaded rectangular areas



Area of k^{th} rectangle = $f(x_k^*) \cdot \Delta x_k$

Now if we let $\max \Delta x_k \to 0$ the width of EVERY rectangle tends to 0 because none of them exceeds the max. So we have the area under the curve now defined more generally as

$$A = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

DEFINITION 5.6.1
(Area under a curve)

If the function f is continuous an [a, b] and if $f(x) \geq 0$ for all x in [a, b], then the area under the curve $y = f(x)$ over the interval [a, b] is defined by $A = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$

Definite Integral of continuous functions with nonnegative values

The limit in the definition we just saw is VERY important. It has special notation. We write it as

$$\int_a^b f(x)dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

The expression on the left is called the definite integral of f from a to b. a and b are called the upper and lower limits of integration respectively.

There is a relationship btw this definite integral and the indefinite integral we talked about earlier.

We will see it later.

So with this notation we can say that

$\int_a^b f(x)dx =$ Area under the curve $y = f(x)$ over [a, b].

Our goal will be to evaluate the definite integral efficiently instead of using the definition all the time.

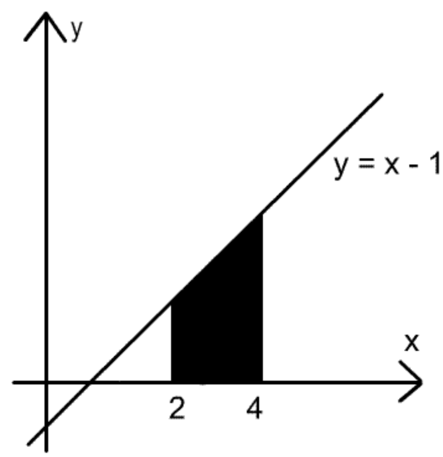
Example

$$\int_2^4 (x-1)dx$$

This represents the area under the curve y = (x – 1) over [2,4].

Here is a picture of it.

Note that the region described



here is just a trapezoid with h = 2, and b_1 = 1 and b_2 = 3. So we can evaluate this integral from basic geometry as $\int_2^4 (x-1)dx = \frac{1}{2}(2)(1+3) = 4$

The sum $\sum_{k=1}^{n} f(x_k^*)\Delta x_k$ is called the Riemann Sum in honor of the German mathematician Bernhard Riemann.

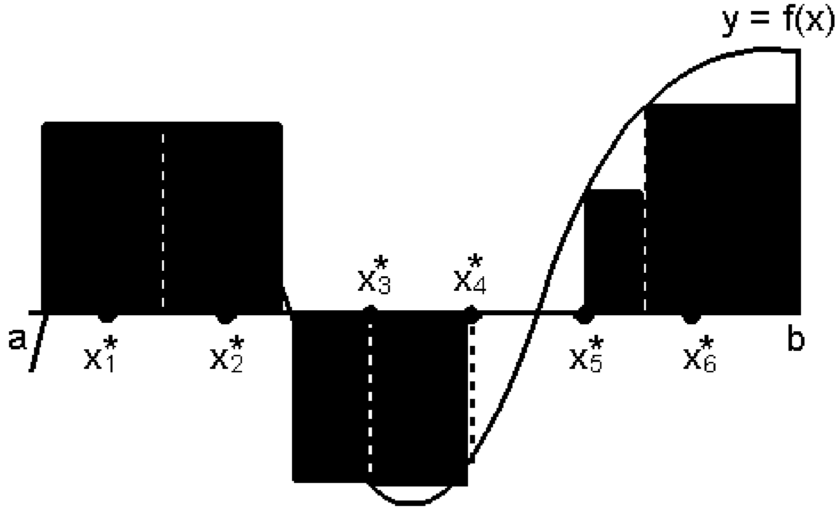
The formula for the area we just saw was for continuous and nonnegative functions. It involves a limit. The question is: does the limit always exist, so that we can always find the area ?

It is proved in advanced levels that for continuous and nonnegative functions, this limit always exists.

Definite Integral of continuous functions with negative and positive values

Now we want to extend our area definition to include continuous functions on [a,b] that have both positive and negative values on [a,b].

Look at the following figure



The rectangles below [a,b] but above the curve f(x) have some area just the way their counterparts above [a,b] but below curve f(x) do. The difference is that we can view the number representing the area of the rectangles below as negative values of f(x) for some x_k^*. So the Riemann sum gives

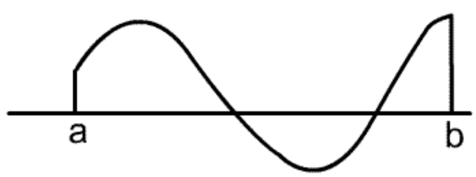
$$\sum_{k=1}^{6} f(x_k^*)\Delta x_k = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \ldots f(x_6^*)\Delta x_6$$

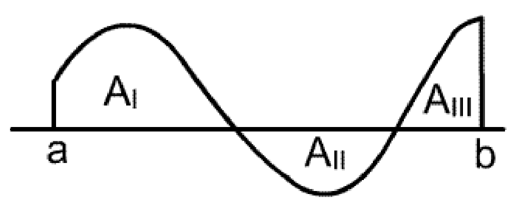
$$= A_1 + A_2 - A_3 - A_4 + A_5 + A_6$$

$$= (A_1 + A_2 + A_5 + A_6) - (A_3 + A_4)$$

This says that the Riemann sum is the difference of two areas: the total area of rectangle above the x-axis, minus the total areas of rectangles below the x-axis.

Now as max delta x goes to 0, the situation still works out nicely with the large # of rectangles filling in any left over space between the earlier rectangles and the curve.





So we have the following definition

Definition 5.6.2

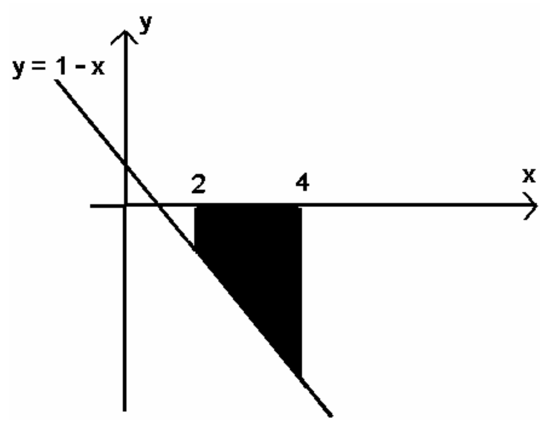
If the function f is continuous on [a,b], and can assume both positive and negative values ,then the **net signed area** A between y= f (x) and the interval [a,b] is defined by

$$\int_a^b f(x)dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^{*})\Delta x_k$$

Net signed area means that the total difference of the two areas above and below may be negative. If that happens then we just treat the negative as representing the fact that the area below is larger than the area above.

EXAMPLE

$$\int_2^4 (1-x)dx$$



Geometrically, this region lies below the interval [2,4]. The region is a trapezoid and we can find the area as 4 using its dimensions. Since its below the x-axis we write

$$\int_2^4 (1-x)dx = -4$$

Definite Integral of functions with discontinuities

For the two definitions we have seen so far, *f* was a continuous function. This guaranteed that the are defined as a limits could always be found since the limits existed for continuous *f*.

If *f* is not continuous, then the area may or may not be found as the limit of the Sum may or may not exist.

Here is a definition for such a function for the area.

DEFINITION 5.6.3
(Area under a curve)

If the function f is defined on the close interval [a, b] then, f is called Riemann integrable on [a, b] or more simply integrable on [a, b] if the limit exists

$\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^{*})\Delta x_k$ If f is integrable on [a, b], then we define The definite integral of f from a to b by

$$\int_a^b f(x)dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^{*})\Delta x_k$$

So far we assumed that the upper limit of integration was greater than the lower limit.
Here is a definition that allows for the limits to be the same, and the case where the upper may be less than the lower.

DEFINITION 5.6.4

(a) If a is in the domain of f , we define

$$\int_a^a f(x)dx = 0$$

If f is integrable on [a, b], then we define

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

Properties of the definite Integral

Here are some properties of the definite integral

Theorem 5.6.5

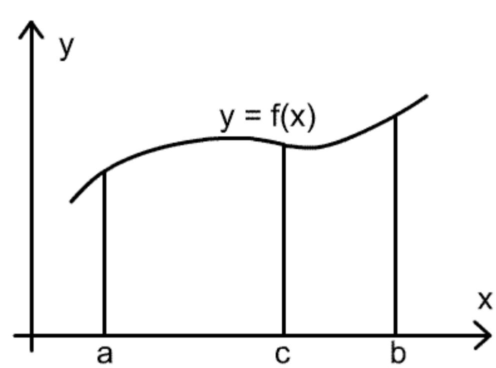
If f and g are integrable on [a, b] and if c is a constant, then cf, f + g, and f – g are integrable on [a, b] and

(a) $\int_a^b cf(x)dx = c\int_a^b f(x)dx$

(b) $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

(c) $\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$

Now look at this figure
It is clear from the picture
that the area under the curve
over [a,b] can be split into
a sum of two areas, one area
from a to c, and the other area from c to b. Formally



Theorem 5.6.5

If f is integrable on a closed interval containing the three points a, b and c then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

no matter how the points are ordered.

Example

$$\int_1^5 f(x)dx = -1, \quad \int_3^5 f(x)dx = 3, \quad \int_3^5 g(x)dx = 4$$

Find

$$\int_1^3 f(x)dx$$

From thm5.6.6 with a = 1, b = 5, c = 3

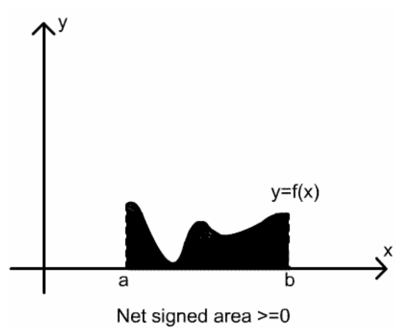
$$\int_1^5 f(x)dx = \int_1^3 f(x)dx + \int_3^5 f(x)dx$$

So

$$\int_1^3 f(x)dx = \int_1^5 f(x)dx - \int_3^5 f(x)dx = -1 - 3 = -4$$

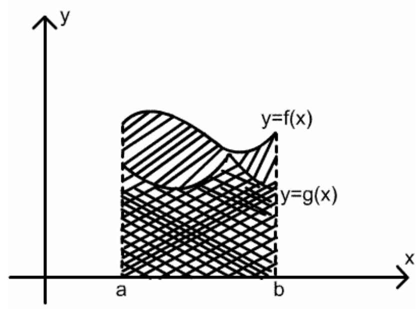
Inequalities involving definite integral

Look at this figure



This says that if a function is nonnegative (the graph does not go below x-axis) on [a,b], then the net area between the curve and x-axis must be greater than or equal to 0.

Look at this figure



area under f >=area g(x)

The graph of f does not go below that of g,

or in other words f(x) >=g(x) over the interval and f and g are nonnegative, then area under f must be >= area under g over the interval

Theorem 5.6.7

(a) If *f* is integrable on [a,b] and $f(x) \geq 0$ for all x in [a,b],then

$$\int_a^b f(x) \geq 0$$

(b) If f and g are integrable functions on[a,b] and $f(x) \geq g(x)$ for all x in

[a, b] then $\int_a^b f(x) \geq \int_a^b g(x)$

Works with <, > <= also

Example

Show that $\int_0^1 \frac{\cos(x)}{2x^3 - 5} dx$ is negative.

On the interval [0,1], cos(x) > 0, but $2x^3 - 5 < 0$. So f(x) as a whole is < 0. So from theorem 5.6.7a), with < instead of >= the integral is negative

Definition 5.6.8

A function f is said to be bounded on interval [a, b] if there is a positive number M such that

$$-M \leq f(x) \leq M$$

for all x in [a, b]. Geometrically, this means that the graph of f on the interval [a, b] lies between the lines y = -M and y = M.

$y = x^2$ on the interval [-2, 2] is bounded since its graph lies between the lines y = 0 and y = 5

Theorem 5.6.9

Let f be a function that is defined at all the points in the interval [a,b].
(a) If f is continuous on [a,b] ,then f is integrable on [a,b].
(b) If f is bounded on [a,b] and has only finite many points of discontinuity on [a,b] then f is integrable on [a,b].
(c) If f is not bounded on [a,b] ,then f is not integrable on [a,b]