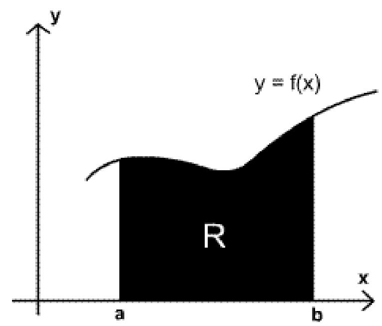
Lecture # 28

Area as Limits

- Definition of Area
- Some technical considerations
- Numerical approx of area
- Look at the figure below



In this figure, there is a region bounded below by the x-axis, on the sides by the lines x= a and x = b, and above by a curve or the graph of a continuous function y = f(x) which is also non-negative on the interval [a, b].

Earlier we saw that such an area can be computed using anti-derivatives.

Let's make the concept precise.

Recall that the slope of the tangent line was defined in terms of the limit of the slopes of secant lines.

Similarly, we will define area of a region R as limits of the areas of simpler regions whose areas are known.

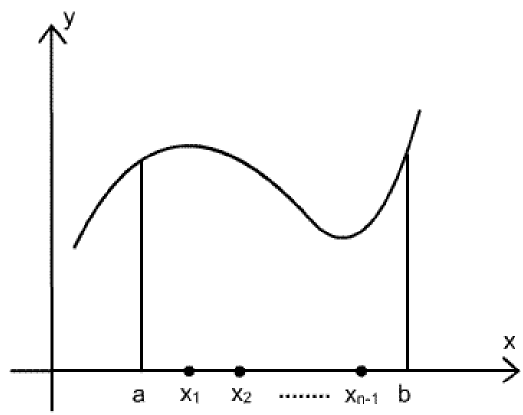
We will break up R into rectangles, and then find the area of each rectangle in R and then add all these areas up.

The result will be an approximation to the region R. Let's call it

If we let n go to infinity, the resulting will give a better and better approximation to R as the rectangle in R will get thinner and thinner, and the gaps will be filled in.

Here is the formal idea

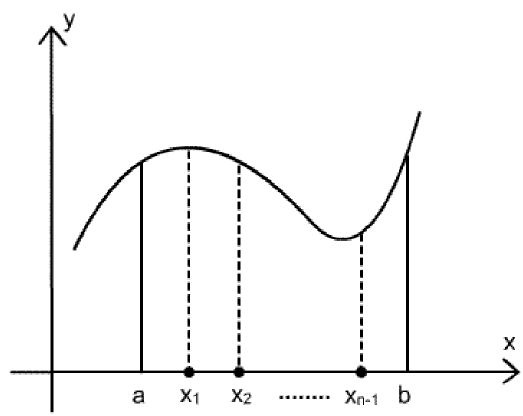
Choose an arbitrary positive integer *n*, and divide the interval [a,b] into *n* subintervals of width $\frac{b-a}{n}$ by inserting *n*-1 equally spaced points between *a* and *b* say $x_1, x_2, \ldots x_{n-1}$



These points of subdivision form a regular partition of [a,b] $a, x_1, x_2, \ldots x_{n-1}, b$

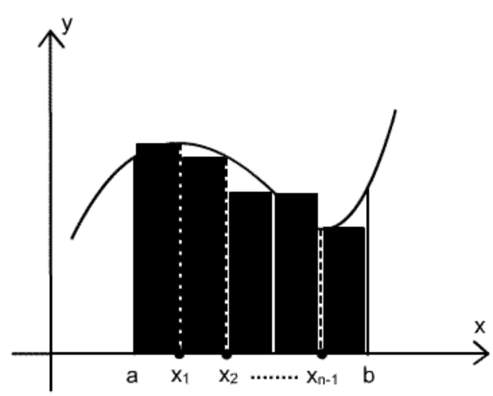
Next draw a vertical line through the points $x_1^*, x_2^*, \ldots x_n^*$

This will divide the region R into *n* strips of uniform width



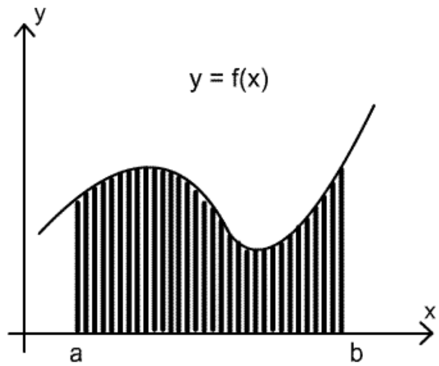
Now we want to approximate the area of each strip by the area of a rectangle. For this, choose an arbitrary point in each subinterval ,

Over each subinterval construct a rectangle whose height is the values of the function *f* at the selected arbitrary point.



The union of these rectangles form the region R_n that we can regard as a reasonable approx to the region R. The area of R_n can be got by adding the areas of all the rectangles forming it

If now we let n get big, the number of rectangles gets big, and the gaps btw the curve and the rectangles get filled in.



As n goes to infinity, the approx gets as good as the real thing.

So we define $A = area(R) = \lim_{n \to +\infty} [area(R_n)]$

For all the following work and computation, we will treat *n* as a POSITIVE integer

For computational purposes, we can write **A** in a different form as follows

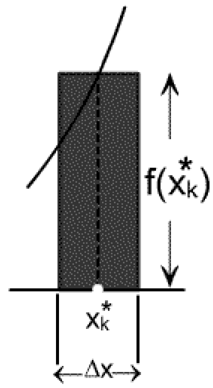
In the interval [a,b], each of the approximation rectangles has width $\dfrac{b-a}{n}$

We call this delta x or $\Delta x = \dfrac{b-a}{n}$

The heights of the approximating rectangles are at the points $x_1^*, x_2^*, \ldots x_n^*$

So the approximating rectangles making up the region have areas

$$f(x_1^*)\Delta x, f(x_2^*)\Delta x, \ldots, f(x_n^*)\Delta x$$



Area of k[th] rectangle = $f(x_k^*)\,.\,\Delta x$

So the total area of R_n is given by

$$area(R_n) = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \ldots + f(x_n^*)\Delta x$$

$$OR$$

$$area(R_n) = \sum_{k=1}^{n} f(x_k^*)\Delta x$$

So now **A** can be written as

$$A = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

This we will take to be the PRECISE def of the area of the region R

Some Technical Considerations

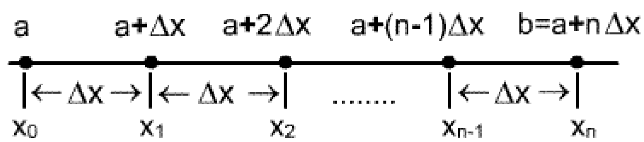
The points $x_1^*, x_2^*, \ldots x_n^*$ were chosen arbitrarily.

What if some other points were chosen? Would the resulting values for f(x) at these points be different? And if so then the definition of area will not be well-defined?

It is proved in advanced courses that since *f* is continuous it doesn't matter what points are taken in a subinterval.

Usually the point x_k^* in a subinterval is chosen so that it is the left end point of the interval, the right end point of the interval or the midpoint of the interval.

Note that we divided the interval [a,b] by the points $x_1, x_2, \ldots, x_{n-1}$ with $x_0 = a$ and $x_n = b$ into equal parts of width Δx



This figure shows that

$$x_k = a + k\Delta x \ \text{ for } \ k = 0,1,2 \ldots n$$

SO

$$x_k^* = x_{k-1} = a + (k-1)\Delta x \qquad \text{Left end point}$$

$$x_k^* = x_k = a + k\Delta x \qquad \text{Right end point}$$

$$x_k^* = \frac{1}{2}(x_{k-1} + x_k) = a + (k - \frac{1}{2})\Delta x \qquad \text{Midpoint}$$

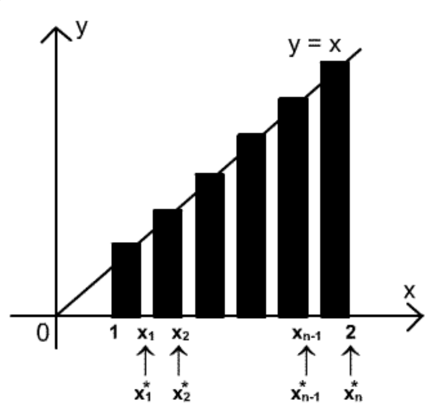
Example

Use the definition of the Area with x_k^* as the right end point of each subinterval to find the area under the line y = x over the interval [1,2].

Subdivide [1,2] into *n* equal parts, then each part will have length

$\Delta x = \dfrac{b-a}{n} = \dfrac{2-1}{n} = \dfrac{1}{n}$ We are taking x_k^* as the right end point so we

Have $x_k^* = a + k\Delta x = 1 + \dfrac{k}{n}$



Thus, the kth rectangle has area

$$f(x_k^*)\Delta x = x_k^*\Delta x = \left(1+\frac{k}{n}\right)\Delta x = \left(1+\frac{k}{n}\right)\frac{1}{n}$$

And the sum of the areas of the *n* rectangles is

$$A = \lim_{n\to+\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x = \lim_{n\to+\infty}\left(\frac{3}{2}+\frac{1}{2n}\right) = \frac{3}{2}$$

Note that the region we have computed the area of is a trapezoid with height h = 1 and bases $b_1 = 1$ and $b_2 = 2$. From basic geometry we have that

Area of trapezoid = $A = \frac{1}{2}h(b_1+b_2) = \frac{1}{2}(1)(1+2) = \frac{3}{2}$

Example

Same problem as before but with left end points.[1,2].

Use the definition of the Area with x_k^* as the left end point of each subinterval to find the area under the line y = x over the interval [1,2].

Subdivide [1,2] into *n* equal parts, then each part will have length

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$$

We are taking x_k^* as the left end point so we

Have $x_k^* = a + k\Delta x = 1 + \frac{k}{n}$

Thus, the kth rectangle has area

$$f(x_k^*)\Delta x = x_k^*\Delta x = \left(1+\frac{k}{n}\right)\Delta x = \left(1+\frac{k}{n}\right)\frac{1}{n}$$

And the sum of the areas of the *n* rectangles is

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n}\left[\left(1+\frac{k}{n}\right)\frac{1}{n}\right] = \sum_{k=1}^{n}\left[\left(\frac{1}{n}+\frac{k}{n^2}\right)\right] = \frac{1}{n}\sum_{k=1}^{n}1 + \frac{1}{n^2}\sum_{k=1}^{n}k$$

$$= \frac{1}{n}\cdot n + \frac{1}{n^2}\left[\frac{1}{2}n(n+1)\right]$$

Note that the region we have computed the area of is a trapezoid with height h = 1 and bases $b_1 = 1$ and $b_2 = 2$. From basic geometry we have that

Area of trapezoid = A = $\frac{1}{2}h(b_1+b_2)=\frac{1}{2}(1)(1+2)=\frac{3}{2}$

Example

Use the area definition with right endpoints of each subinterval to find the area under the parabola $y=9-x^2$ over the interval [0,3].

Do calculations from book on page 272 (end) and page 273 for example.

Numerical Approximations of Area

As we have all seen so far, the computations involved in computing the limits are tedious and long. In some cases, it is even impossible to carry out the computations by the definition of the area. In such cases, it is easier to get a GOOD approx for the area using large values of *n* and using a computer or a calculator.

Example

Use a computer of a calculator to find the area under the curve y = $9-x^2$ over the interval [0,3] and *n* = 10, 20 and 50.

Left end point approximation			Right end point approximation		Mid point approximation	
K	x_k^*	$9-(x_k)^2$	x_k^*	$9-(x_k)^2$	x_k^*	$9-(x_k)^2$
1	0.0	9.000000	0.3	8.910000	0.15	8.977500
2	0.3	8.910000	0.6	8.640000	0.45	8.797500
3	0.6	8.640000	0.9	8.190000	0.75	8.437500
4	0.9	8.190000	1.2	7.560000	1.05	7.897500
5	1.2	7.560000	1.5	6.750000	1.35	7.177500
6	1.5	6.750000	1.8	5.760000	1.65	6.277500
7	1.8	5.760000	2.1	4.590000	1.95	5.197500
8	2.1	4.590000	2.4	3.240000	2.25	3.937500
9	2.4	3.240000	2.7	1.710000	2.55	2.497500
10	2.7	1.710000	3.0	0.00000	2.85	0.877500
$\Delta x\sum_{k=1}^{n} f(x_k^*)$ = 19.30500			=16.605000		=18.022500	

And for different values of n we have the following approximations.

n	**Left end point approximation**	**Right end point approximation**	**Mid point approximation**
10	19.305000	16.605000	18.02250
20	18.663750	17.313750	18.005625
50	18.268200	17/728200	18.000900