Lecture #. 27

Sigma Notation

- Sigma notation is used to write lengthy sums in compact form.
- Sigma or Σ is an Upper case letter in Greek.
- This symbol is called Sigma or summation as it is used to represent lengthy sums.

Here is an example of how this notation works.

Consider the sum

$$1^2+2^2+3^2+4^2+5^2$$

Note that every term in this sum is the square of an integer from 1 to 5.

Let's assign these integers a variable *k* and keep it in mind that this *k* can take on values from 1 to 5.

Then we can say that k^2 represent each of the elements in the sum. So we can write

this as $\sum_{k=1}^{5} k^2$

"Summation of k^2 where k goes from 1 to 5

Example

$$\sum_{k=4}^{8} k^3 = 4^3+5^3+6^3+7^3+8^3$$

$$\sum_{k=1}^{5} 2k = 2(1)+2(2)+2(3)+2(4)+2(5) = 2+4+6+8+10$$

$$\sum_{k=0}^{5} (-1)^k (2k+1) = 1-3+5-7+9-11$$

The number on the top are called the upper limits of the summation and the numbers at the bottom are called lower limits of the summation.

The letter *k* is called the index of the summation

It is not necessary that the letter *k* represents the index of summation. We could use *i or j or m etc.*

Example

$$\sum_{i=1}^{4} \frac{1}{i}, \sum_{n=1}^{4} \frac{1}{n}, \sum_{j=1}^{4} \frac{1}{j},$$

All denote the sum

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$$

If the upper and lower limits are the same, then the summation reduces to just one

term $\sum_{k=2}^{2} k^3 = 2^3$

If the expression to the right of the summation does not involve the index of the summation, then do the following

$$\sum_{i=1}^{5} 2 = 2+2+2+2+2$$

$$\sum_{k=3}^{6} x^3 = x^3+x^3+x^3+x^3$$

A sum can be written in more than one way with the Sigma notation if we change the limits of the summation

Example

The following summations all represent the sum of the first five positive integers

$$\sum_{k=1}^{5} 2k = 2+4+6+8+10$$

$$\sum_{k=0}^{4} (2k+2) = 2+4+6+8+10$$

$$\sum_{k=2}^{6} (2k-2) = 2+4+6+8+10$$

Changing the index of the summation

It is often necessary and useful to change a given Sigma notation for a sum to another sigma notation with different limits of summation

Example

Express $\sum_{k=3}^{7} 5^{k-2}$ in sigma notation so that the lower limit is 0 rather than 3.

Define a new summation index *j* by the following formula

$j = k-3 \Rightarrow k = j+3$

Then as *k* runs from 3 to 7, *j* runs from 0 to 4. So

$$\sum_{k=3}^{7} 5^{k-2} = \sum_{j=0}^{4} j^{(j+3)-2} = \sum_{j=0}^{4} j^{j+1}$$

You should check that the two actually represent the same sum by putting values into the index in both notations.

To represent a general sum, we will use letters with subscripts.

Example

$a_1 + a_2 + a_3$ represent the general sum with three terms

This can also be written as in sigma notation

$$\sum_{k=1}^{3} a_k = a_1 + a_2 + a_3$$

General sum with n terms can be written as

$$\sum_{k=1}^{n} b_k = b_1 + b_2 + b_3 + \ldots + b_n$$

Properties of Sigma notation

Here are a few properties of sigma notation that will be helpful later on

THEOREM 5.4.1

a) $\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$

b) $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

c) $\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

Here are some sum formulas written in sigma notation that will be helpful later.

THEOREM 5.4.2

$$a)\ \sum_{k=1}^{n} k = 1+2+3+\ldots\ldots+n = \frac{n(n+1)}{2}$$

$$b)\ \sum_{k=1}^{n} k^2 = 1^2+2^2+3^2+\ldots\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$c)\ \sum_{k=1}^{n} k^3 = 1^3+2^3+3^3+\ldots\ldots+n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Example

$$\sum_{k=1}^{30} k(k+1) = \sum_{k=1}^{30}(k^2+k) = \sum_{k=1}^{30} k^2 + \sum_{k=1}^{30} k$$

$$= \frac{(30)(31)(61)}{6} + \frac{30(31)}{2} = 9920$$

$$\uparrow$$

Theorem 5.4.2 a) and b)

In a formula involving summation like this one

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+2)}{6}$$

$$\Rightarrow 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+2)}{6}$$

The left part is called the open form of the sum.

The right part is called the closed form of the sum.