**Lecture # 26**

**Integration by substitution**

This is like the Chain Rule we saw for differentiation. The idea is to integrate functions that are composition of different functions.

$$\frac{d}{du}\left[G(u)\right] = f(u)$$

Also this means that

$$\int f(u)du = \int \frac{d}{du}\left[G(u)\right]du = G(u) + C$$

Let $u$ be a function of $x$. Then we have

$$\frac{d}{dx}\left[G(u)\right] = \frac{d}{du}\left[G(u)\right]\cdot\frac{du}{dx} = f(u)\frac{du}{dx}$$

Now if we apply the integral on both sides w . r . t $x$ we get

$$\int\left( f(u)\frac{du}{dx}\right) dx = \int \frac{d}{dx}\left[G(u)\right]\ dx = G(u) + C$$

$\Rightarrow$

$$\int\left( f(u)\frac{du}{dx}\right) dx = \int f(u)du$$

**Example**

$$\int (x^2+1)^{50}.2x\ dx$$

Let $u = x^2+1$. Then $\dfrac{du}{dx} = 2x$.

Now we can rewrite the given problem as

$$\int (x^2+1)^{50}.2x\ dx = \int\left[u^{50}\frac{du}{dx}\right]dx = \int u^{50}du$$

$$\uparrow \qquad\qquad \uparrow$$

$$\int\left[ f(u)\frac{du}{dx}\right] dx = \int f(u)du$$

$$\int u^{50} du = \frac{u^{51}}{51} + C = \frac{(x^2+1)^{51}}{51} + C \quad , \text{ where } u = x^2 + 1$$

Caution: Don't feel tempted to just add 1 to the power 50 in the original problem!! That will be incorrect.

The reason is that the "Power Rule" for the integrals is applicable to function which are not a composition of others.

In this example, $(x^2+1)^{50}$ is a composition of two functions.

Here is a summary of the **general procedure** we need to follow to do integration by ***u*-substitution**.

**Step 1:** Make the choice for *u*, say *u* = g (x)

**Step 2:** Compute du/dx = g/(x)

**Step 3:** Make the substitution u = g (x), du = g/(x) dx in the original integral.

By this point, the whole original integral should be in terms of u and there should be no x's in it.

**Step 4:** Evaluate the resulting integral

**Step 5:** Replace u by g (x) so the final answer is in *x*.

- No hard and fast rule for choosing *u*.
- The choice of u should be such that the resulting calculus
- Results in a simplified integral.
- There may be more than one choice of *u* in some cases.

Only practice and experience makes things easier.

Think of this choice of *u* like playing chess. You need to choose it so that the future looks bright for problem solution!

Integrand is the derivative of a known function with a constant added or subtracted from the independent variable.

If the integrand is the derivative of a known function with a constant added or subtracted from the independent variable, then the substitution is the easiest.

Here is table 5.2.1 that we saw earlier. This will help recall BASIC integration formulas.

| DIFFERENTIATION FORMULA | INTEGRATION FORMULA |
|---|---|
| $\frac{d}{dx}[x] = 1$ | $\int dx = x + C$ |
| $\frac{d}{dx}\left[\frac{x^{r+1}}{r+1}\right] = x^r$ | $\int x^r dx = \frac{x^{r+1}}{r+1} + C (r \neq -1)$ |
| $\frac{d}{dx}[\sin x] = \cos x$ | $\int \cos x dx = \sin x + C$ |
| $\frac{d}{dx}[-\cos x] = \sin x$ | $\int \sin x dx = -\cos x + C$ |

| DIFFERENTIATION FORMULA | INTEGRATION FORMULA |
|---|---|
| $\frac{d}{dx}[\tan x] = \sec^2 x$ | $\int \sec^2 x dx = \tan x + C$ |
| $\frac{d}{dx}[-\cot x] = \csc^2 x$ | $\int \csc^2 x dx = -\cot x + C$ |
| $\frac{d}{dx}[\sec x] = \sec x \tan x$ | $\int \sec x \tan x dx = \sec x + C$ |
| $\frac{d}{dx}[-\csc x] = \csc x \cot x$ | $\int \csc x \cot x dx = -\csc x + C$ |

We want to integrate a function in which "The integrand is the derivative of a known function with a constant added or subtracted from the independent variable"

**Example**

$$\int \sin(x+9)dx = \int \sin(u)du = -\cos(u) + C = -\cos(x+9) + C$$

$$\uparrow$$

$$u = x+9$$

$$\frac{du}{dx} = 1 \quad \Rightarrow \quad du = 1.dx$$

The integrand sine is the derivative of cosine function

**Example**

$$\int (x-8)^{23}dx = \int u^{23}du = \frac{u^{24}}{24} + C = \frac{(x-8)^{24}}{24} + C$$

$$\uparrow$$

$$u = x-8 \quad \Rightarrow \quad du = 1.dx = dx$$

Integrand is the derivative of a known function and a constant multiplies the independent variable

**Example**

$$\int \cos(5x)dx = \int \cos(u)\frac{du}{5} = \frac{1}{5}\int \cos(u)du = \frac{1}{5}\sin(u) + C = \frac{1}{5}\sin(5x) + C$$

$$\uparrow$$

$$u = 5x$$

$$du = 5dx \quad \Rightarrow \quad \frac{du}{5} = dx$$

**Example**

$$\int \sin^2(x)\cos(x)dx$$

Let $u = \sin(x),$ then $du = \cos(x)dx$

So

$$\int \sin^2(x)\cos(x)dx = \int u^2du = \frac{u^3}{3} + C = \frac{\sin^3(x)}{3} + C$$

**Example**

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$$

Let $u=\sqrt{x},$ then $du=\frac{1}{2\sqrt{x}}dx$

So

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx=\int 2\cos(u)du=2\int\cos(u)du=2\sin(u)+C=2\sin(\sqrt{x})+C$$

Complicated **example**

$$\int t^4\sqrt[3]{3-5t^5}dt$$

What should the substitution be?

Well the idea is to get an expression to equal $u$ so that when we differentiate it, we get a formula that involves a $du$ and everything that was left over in $x$ after the substitution.

$$\int t^4\sqrt[3]{3-5t^5}dt$$

$$u=3-5t^5$$

$$du=-25t^4dt \quad\Rightarrow\quad -\frac{1}{25}du=t^4dt$$

So

$$\int t^4\sqrt[3]{3-5t^5}dt=-\frac{1}{25}\int\sqrt[3]{u}\;du=-\frac{1}{25}\int u^{\frac{1}{3}}\;du=-\frac{1}{25}\frac{u^{\frac{4}{3}}}{\frac{4}{3}}+C$$

$$=-\frac{3}{100}(3-5t^5)^{\frac{4}{3}}+C$$