MTH101 Solution: Practice Questions
Lecture No. 26 to 28

Question 1: Evaluate the integral by using substitution method: $\int \frac{1}{t^2}\cos\left(\frac{1}{t}-1\right)dt$.

Answer: $-\sin\left(\frac{1}{t}-1\right)+C$

Solution:

Let $u=\frac{1}{t}-1,$

$\Rightarrow \quad du=-\frac{1}{t^2}dt \quad \Rightarrow \frac{1}{t^2}dt=-du,$

$$\therefore \int \frac{1}{t^2}\cos\left(\frac{1}{t}-1\right)dt=\int \cos u\,(-du),$$

$$=-\sin u+C,$$

$$=-\sin\left(\frac{1}{t}-1\right)+C. \qquad (\because \text{by replacing with the original value})$$

Question 2: Evaluate the indefinite integral by substitution method: $\int \frac{(1+\ln x)^3}{x}dx$.

Answer: $\frac{(1+\ln x)^4}{4}+C$

Solution:

Let $t=1+\ln x,$

$\Rightarrow \quad dt=\frac{1}{x}dx,$

Hence, $\int \frac{(1+\ln x)^3}{x}dx=\int t^3dt=\frac{t^4}{4}+C=\frac{(1+\ln x)^4}{4}+C.$

Question 3: Evaluate the sum: $\sum_{k=1}^{7}(k^2-6).$

Answer: 98

Solution:

$$\sum_{k=1}^{7}(k^2-6)=\sum_{k=1}^{7}k^2-\sum_{k=1}^{7}6\ ,$$
$$=\frac{7(7+1)(2(7)+1)}{6}-6(7),$$
$$=\frac{840}{6}-42,$$
$$=98.$$

Question 4: Express $\sum_{k=2}^{5}3^{k-2}$ in sigma notation so that the lower limit is '0' rather than '2'.

Answer: $\sum_{j=0}^{3}3^{j}$

Solution:

We will define a new summation index 'j' by the relation
$j = k - 2 \ or \ j + 2 = k,$
Now when $k = 2\ , j = 2 - 2 = 0,$
When $k = 5\ , j = 5 - 2 = 3,$
So the new summation will become $\sum_{j=0}^{3}3^{j+2-2}=\sum_{j=0}^{3}3^{j}$.

Question 5: Find the area of the kth rectangle below the curve $y = x^2$ on the interval $[0,2]$ by taking x_k^* as

i. Right end point
ii. Left end point

Answer: **(i)** $\frac{8k^2}{n^3}$ **(ii)** $\frac{8(k-1)^2}{n^3}$

Solution:

In order to find the area of kth rectangle, first of all we will find the width or base of the rectangle that is Δx .

$$\Delta x=\frac{b-a}{n}=\frac{2}{n}.$$

Right end point

$$x_k^* = a + k\Delta x,$$
$$=0+k.\frac{2}{n}=\frac{2k}{n}.$$

The height of the rectangle is

$$f(x_k^*) = \left(\frac{2k}{n}\right)^2 = \frac{4k^2}{n^2}.$$

Thus, the area of the kth rectangle will be

$$\begin{aligned} \text{Area} &= \text{height} \times \text{base} \\ &= f(x_k^*).\Delta x, \\ &= \frac{4k^2}{n^2}.\frac{2}{n} = \frac{8k^2}{n^3}. \end{aligned}$$

Left end point

$$\begin{aligned} x_k^* &= a + (k-1)\Delta x, \\ &= 0 + (k-1).\frac{2}{n} = \frac{2(k-1)}{n}. \end{aligned}$$

The height of the rectangle is

$$f(x_k^*) = \left(\frac{2(k-1)}{n}\right)^2 = \frac{4(k-1)^2}{n^2}.$$

Thus, the area of the kth rectangle will be

$$\begin{aligned} \text{Area} &= \text{height} \times \text{base} \\ &= f(x_k^*).\Delta x, \\ &= \frac{4(k-1)^2}{n^2}.\frac{2}{n} = \frac{8(k-1)^2}{n^3}. \end{aligned}$$

Question 6:
Find the approximate area under the graph of function $y = x$ over the interval $[0, 2]$ by taking $\Delta x = \frac{2}{n}$ and $x_k^* = \frac{2k}{n}$.

Answer: 2

Solution:

Given that $\Delta x = \frac{2}{n}$ and $x_k^* = \frac{2k}{n}$,

$$f(x_k^*) = \frac{2k}{n},$$

$$f(x_k^*).\Delta x = \frac{2k}{n}.\frac{2}{n} = \frac{4k}{n^2},$$

$$\sum_{k=1}^{n} f(\mathrm{x}_k^*).\Delta\,\mathrm{x} = \sum_{k=1}^{n} \frac{4k}{n^2},$$

$$= \frac{4}{n^2}\sum_{k=1}^{n} k,$$

$$= \frac{4}{n^2}\frac{n(\mathrm{n}+1)}{2} = \frac{2(\mathrm{n}+1)}{n} = 2\left(1+\frac{1}{n}\right).$$

$$\text{Area} = \lim_{n\to\infty}\left(\sum_{k=1}^{n} f(\mathrm{x}_k^*).\Delta\,\mathrm{x}\right) = \lim_{n\to\infty} 2\left(1+\frac{1}{n}\right) = 2.$$