Lecture # 25

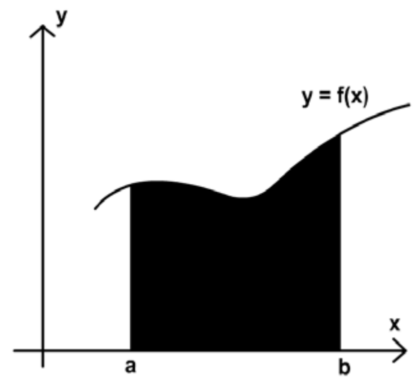
Integrations

In this lecture we will look at the beginnings of the other major Calculus problem.

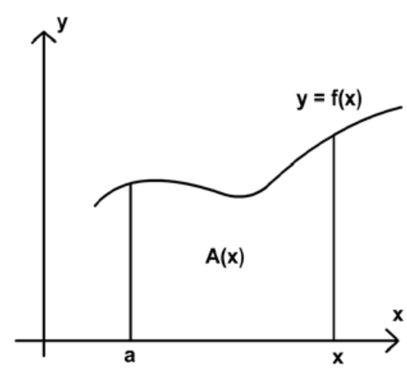
- The Area Problem
- Anti-derivatives (Integration)
- Integration formulas
- Indefinite Integral
- Properties of Indefinite Integral

The Area Problem

Given a continuous and non negative function on an interval [a , b], find the area between the graph of *f* and the interval [a , b] on the x-axis.

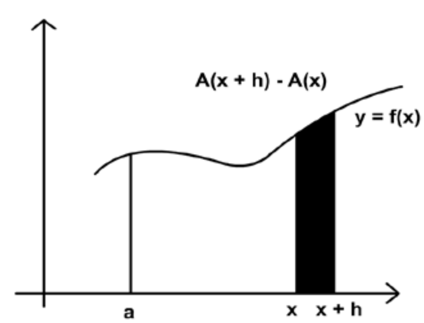


Instead of trying to solve a particular case like the one in the picture we just saw, we will generalize to solve this problem where the right end point will be any number *x* greater than or equal to *b* instead of just b.
We will denote the area we are trying to find as A(x) because this will be a function of *x* now as it depends on how far away *x* is from *a*.



It was the idea of Newton and Leibniz that to find the unknown area A(x), first find it derivative $A'(x)$ and use this derivative to determine what A (x) is ! Interesting approach.
So we want to find out first.

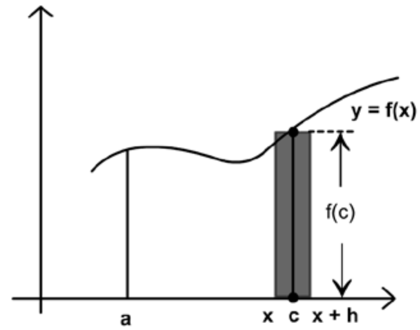
$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$$



Let's assume for now that h > 0
The top of the derivative quotient is the difference of the two areas A(x) and A(x + h)
Let c be the midpoint of between x and x +h. Then the difference of areas can be approximated by the area of the rectangle with base length *h* and height *f* (c).

So we have

$$\frac{A(x+h)-A(x)}{h} \approx \frac{f(c)\cdot h}{h} = f(c)$$



Note that the error in the approximation from this rectangle in **A** will approach 0 as h goes to 0.
Then we have

$$A'(x) = \lim_{h\to 0}\frac{A(x+h)-A(x)}{h} = \lim_{h\to 0} f(c)$$

As h goes to 0, c approaches x. Also, *f* is assumed to be a continuous function, so we have that f(c) goes to f (x) as c goes to *x*. Thus

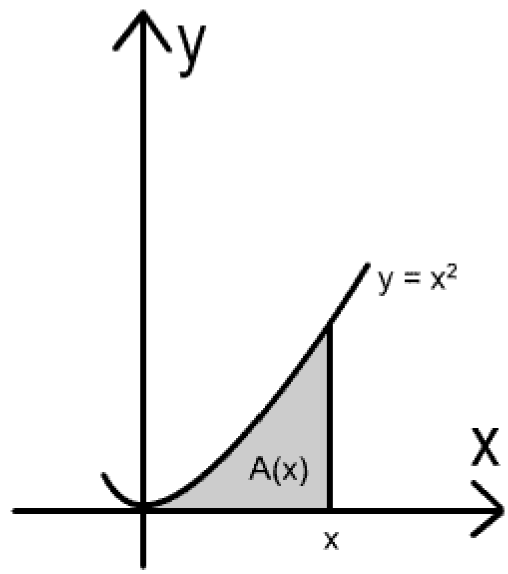
$$\lim_{h\to 0} f(c) = f(x) \quad \Rightarrow \quad A'(x) = f(x)$$

So: The derivative of the area function A (x) is the function whose graph forms the upper boundary of the region under which the area is to be found

Example

Find the area of the region under the graph of $y = f(x) = x^2$ over the interval [0 , 1]
Look at the situation over the interval [0 , x]. Then we have from the discussion that $A'(x) = x^2$
To find A(x) we look for a function whose derivative is x^2
This is called an antidifferentiation problem as we are trying to find A(x) by undoing a differentiation.
A guess is the function

$$A(x) = \frac{1}{3}x^3$$



This is a formula for the areas function. So on the interval [0 , 1], we have x = 1 and our result is
A(1) = 1/3 units

Anti-derivatives
Definition 5.2.1
A function F is called antiderivative of a function *f* on a given interval if $F'(x) = f(x)$ for all x in the interval.

Example

The functions $\frac{1}{3}x^3, \quad \frac{1}{3}x^3 - \pi, \quad \frac{1}{3}x^3 + C$ are all anti-derivatives of

$f(x) = x^2$ on the interval $(-\infty, +\infty)$

As the derivative of each is $f(x) = x^2$

If F (x) is any anti-derivative of *f* (x), then so is F (x) +C where C is a constant.
Here is a theorem

THEOREM 5.2.2

If F(x) is any antiderivative of f(x) on a given intreval,then for any value of C the function F(x) + C is also an antiderivative of f(x) on that interval; moreover every antiderivative of f(x) on the interval is expressible in the form F(x) + C. where C is constant.

Indefinite Integral

The process of finding anti-derivatives is called anti-differentiation or Integrations.

If there is some function F such that $\frac{d}{dx}[F(x)] = f(x)$

Then function of the form F (x) +C are anti-derivatives of f (x).

We denote this by $\int f(x)dx = F(x) + C$

The symbol $\int$ is called the integral sign and f (x) is called the integrand.

It is read as the "Indefinite integral of f (x) equals F (x)" $\int f(x)dx = F(x) + C$

The right side of the above equation is not a specific function but a whole set of possible functions.
That's why we call it the Indefinite integral.
C is called the constant of integrations.

Example

As we saw earlier, the anti-derivatives of $f(x) = x^2$ are functions of the form So we can write

$F(x) = \frac{1}{3}x^3 + C$

The dx serves to identify the independent variable in the function involved in the integration.

Examples:

DERIVATIVE FORMULA	EQUIVALENT INTEGRATION FORMULA
$\frac{d}{dx}[x^3] = 3x^2$	$\int 3x^2 dx = x^3 + C$
$\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x dx = \sin x + C$
$\frac{d}{dt}[\tan t] = \sec^2 t$	$\int \sec^2 t dt = \tan t + C$
$\frac{d}{dx}[u^{3/2}] = 3/2u^{1/2}$	$\int 3/2u^{1/2} du = u^{3/2} + C$

Example
From the table we just saw, we obtain the following results.

$$\int x^2 dx = \frac{x^3}{3} + C$$

$$\int x^3 dx = \frac{x^4}{4} + C$$

$$\int \frac{1}{x^5} dx = \int x^{-5} dx = -\frac{1}{4x^4} + C$$

Properties of Indefinite Integral

$$\frac{d}{dx}\left[\int f(x)dx\right] = f(x)$$

THEOREM 5.2.3

a) A constant can be moved through an integral sign; that is,

$$\int cf(x)dx = c\int f(x)dx$$

b) An antiderivative of a sum is the sum of the antiderivative; that is

$$\int [f(x) + g(x)dx = \int f(x)dx + \int g(x)dx$$

c) An antiderivative of a difference is the difference of the antiderivative; that is

$$\int [f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$$

Example

Evaluate $\int 4\cos(x)dx$

$$\int 4\cos(x)dx = 4\int \cos(x)dx = 4[\sin(x) + C] = 4\sin(x) + K$$

Where 4C = K

Example

$$\int (x^2 + x)dx$$

$$\int (x^2 + x)dx = \int x^2 dx + \int x \cdot dx = \frac{1}{3}x^3 + \frac{1}{2}x^2 + C$$

Generalized version of the Theorem 5.2.3 b and c

$$\int [c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x)]dx$$

$$= c_1 \int f_1(x)dx + c_2 \int f_2(x)dx + \ldots + c_n \int f_n(x)dx$$

Example

$$\int (3x^6 - 2x^2 + 7x + 1)dx$$

$$= 3\int x^6 dx - 2\int x^2 dx + 7\int x dx + \int dx$$

$$= \frac{3x^7}{7} - \frac{2x^3}{3} + \frac{7x^2}{2} + x + C$$

Example

$$\int \frac{\cos(x)}{\sin^2(x)} dx$$

$$= \int \frac{1}{\sin(x)} \frac{\cos(x)}{\sin(x)} dx$$

$$= \int \cos ec(x)\cot(x)dx = -\cos ec(x) + c$$