Lecture # 24

Newton's Method, Rolle's Theorem, and the Mean Value Theorem

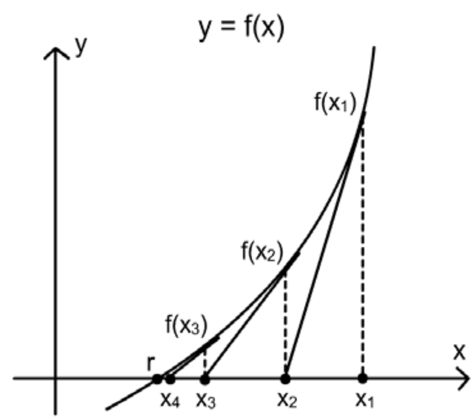
- Newton's method for approximating solutions to f(x)=0
- Some difficulties with Newton's method
- Rolle's theorem
- Mean Value Theorem

Newton's method for approximating solutions to f (x) =0

We have seen in algebra that the solution for the equation $ax+b=0$ is $x=-\dfrac{b}{a}$.

Similarly we have algebraic formulas for polynomial equation up to degree 5.
But there is no algebraic solution to the equation of the kind
$x-\cos(x)=0$
For this equation and many like it, we settle for approximate solutions. How do we approximate the solutions? There are many methods, and one of them is Newton's method. Here is how Newton's method works to find approximate solutions to equations.
What does it mean for an equation to have a solution? or that f (x) = 0 ?
It means that we are looking for those x values, for which the corresponding y value or f (x) is 0.
This means that the solutions are those points where the graph of the function crosses the x axis.
Suppose that x = r is the solution we are looking for.
Let's approximate it by an initial guess called.
We draw a line tangent to the graph of the given function at the point.
If the tangent line is not parallel to the x-axis, then it will x_2 eventually intersect the x-axis at some point x_1 which will generally be closer to *r* than We will repeat the process, with a tangent line at x2 that meets x axis at x3, and so on…
This is Newton's method.

We need a formula for this method.
Note that the tangent line at x_1 has the equation
$y-f(x_1)=f'(x_1)(x_2-x_1)$
If $f'(x_1)\neq 0$, then the line meets the x axis at $(x_2,0)$



Plug this coordinate into above equation, we get
$-f(x_1)=f'(x_1)(x_2-x_1)$
$$\Rightarrow x_2=x_1-\frac{f(x_1)}{f'(x_1)}$$
Repeating this process for a third point $(x_3,0)$ gives
$-f(x_2)=f'(x_2)(x_3-x_2)$
$$\Rightarrow x_3=x_2-\frac{f(x_2)}{f'(x_2)}$$

In general, then we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

There is limiting process involved here for finding the solutions. We get as close as we like to the solution.

Example

The equation $x = \cos(x)$ has a solution between 0 and 1. Approximate it using Newton's method.

Rewrite as $x = \cos(x)$ so our function is $f(x) = x - \cos(x)$. The derivative is

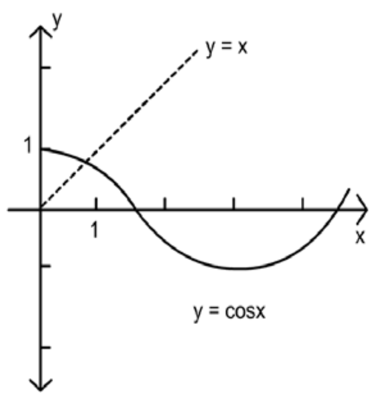
$$f^{/}(x) = 1 + \sin(x)$$

So we have

$$x_{n+1} = x_n - \frac{x_n - \cos(x_n)}{1 + \sin(x_n)}$$

As our approximation formula

Here is a graph of the situation. From the graph it looks like the solution is closer to 1 than 0. So we will use $x_1 = 1$.So we get the following approximations.



$$x_2 = x_1 - \frac{x_1 - \cos(x_1)}{1 + \sin(x_1)} = 1 - \frac{1 - \cos(1)}{1 + \sin(1)} = 0.7503$$

$$x_3 = x_2 - \frac{x_2 - \cos(x_2)}{1 + \sin(x_2)} = 0.7503 - \frac{0.7503 - \cos(0.7503)}{1 + \sin(0.7503)} = 0.7391$$

You may continue if you will, but we will say that the solution is approximately $x \approx 0.7391$

Some difficulties with Newton's method

- Newton's method does not always work.
- If for some values of *n* $f^{/}(x_n)$ then the formula for Newton's method involves division by 0 and we are out of business.

- Such a case will occur if the tangent line for some approximation has slope 0 or is parallel to the x axis.
- Sometimes the approximations don't converge to a solution.

Consider the equation $x^{\frac{1}{3}} = 0$

The only solution is x = 0. Let's approximate it by Newton's Method with initial approx $x_1 = 1$. We get the following Formula

$$x_{n+1} = x_n - \frac{(x_n)^{\frac{1}{3}}}{\frac{1}{3}(x_n)^{-\frac{2}{3}}} = -2x_n$$

Plug in $x_1 = 1$ and then the following approx to see that the values do not converge

Rolle's theorem

Rolle's Theorem says essentially that for a certain kind of function, if it crosses the x-axis at two point, then there is one point between those two points where the derivative of *f* is 0.

THEOREM 4.9.1
(Rolle's Theorem)

Let f be differentiable on (a,b) and continuous on [a,b] . If f(a) = f(b) = 0, then there is at least one point c in (a,b) where $f'(c) = 0$.

Example

The function $f(x) = \sin(x)$ is continuous and differentiable everywhere, hence continuous on [0 , 2π] and differentiable on (0, 2π). Also, f (0) = sin (0) = 0 and
f(2π)=sin (2π) = 0

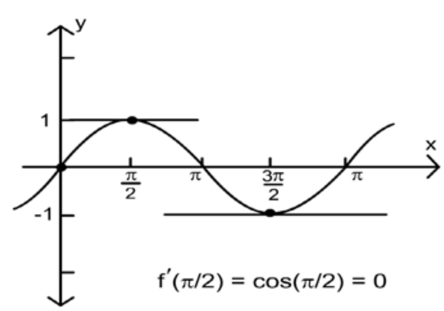
So the function satisfies the hypotheses of Rolle's Theorem. So there exists a point *c* in the interval (0 , 2π) such that

$$f'(c) = \cos(c) = 0$$

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

$$\Rightarrow$$

$$y = \frac{f(b) - f(a)}{b - a}(x - a) + f(a)$$



Here is a more tangible way to think of Roll's theorem

I leave from Lahore to Islamabad.
When i start driving from Lahore, my velocity is 0, and when i reach Islamabad, my velocity is 0 as well.
Velocity is a continuous function on the interval [0 , 376].
Also, velocity is differentiable on (0 , 376) as its derivative acceleration is defined at each point on the velocity curve.
Hence during my drive from Lahore to ISB, there is some point on the motorway where the acceleration of the car was 0.
One could argue: What if you keep accelerating on the motorway!

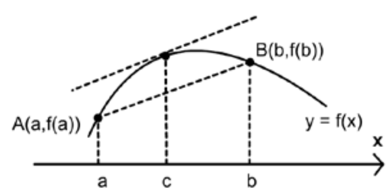
Mean Value Theorem
This says basically that under the right conditions, a function will have the same slope for the tangent line at a point as that of a certain secant line.

THEOREM 4.9.2
(Mean-Value Theorem)

Let f be differentiable on (a,b) and continuous on [a,b). Then there is at least one point c in (a,b) where

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof of MVT
From this figure we have the following:
Slope of Secant line joining A and B:



v(x) = vertical distance between curve f(x) and secant line through A and B

$$v(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}(x - a) + f(a) \right]$$

Since f(x) is continuous [a , b] and differentiable on (a , b), so is v(x) by its formula involving f(x).
Also note that

$v(a) = 0$ and $v(b) = 0$

So v(x) satisfies the assumptions of Roll's theorem on the interval [a , b]. So there is a point c in (a , b) such that v'(c)=0. But note that

$$v'(x) = f'(x) - \left[\frac{f(b) - f(a)}{b - a}\right]$$

$$\Rightarrow v'(c) = f'(c) - \left[\frac{f(b) - f(a)}{b - a}\right]$$

So by this last formula, at the point where $v'(c) = 0$, we have

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$