Lecture # 23

Maximum and Minimum Values of Functions

- Absolute Extrema
- Finding Absolute Extrema for a continuous function
- Summary of extreme behaviors of functions over
- Applied maximum and minimum problems
- Problems involving finite closed intervals
- Problems involving intervals that are non-finite and closed

Absolute Extrema

Previously we talked about relative maxima, relative minima of functions.
These were like the highest mountain and the deepest valley in a given vicinity or neighborhood.
Now we will talk about absolute maximum and minimum values of functions.
These are like the highest peak in a mountain range, and the deepest valley.

DEFINITION 4.6.1

If $f(x_0) \geq f(x)$ for all x in the domain of f, then $f(x_0)$ is called the absolute maximum value or simply the maximum value of f.

DEFINITION 4.6.2

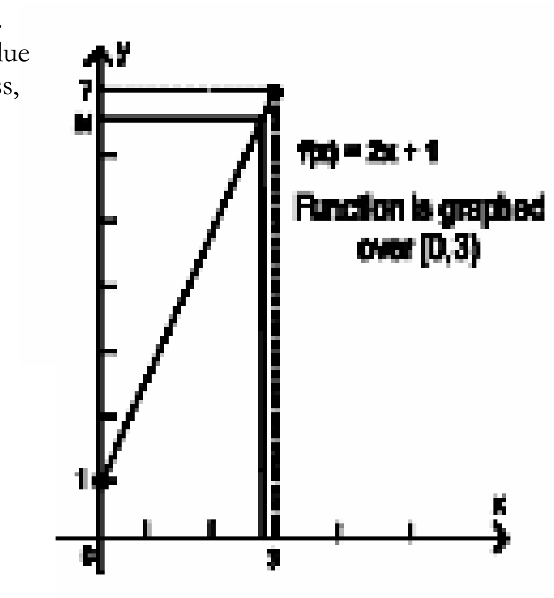
If $f(x_0) \leq f(x)$ for all x in the domain of f, then $f(x_0)$ is called the absolute minimum value or simply the minimum value of f.

Absolute maximum means that the value is the maximum one over the entire domain of the function.
Absolute minimum means that the value is the minimum one over the entire domain of the function.
If we think of the Earth's surface as defining some function, then its absolute maximum will be Mt. Everest, and absolute minimum will be the Marianna Trench in the Pacific Ocean near Hawaii.

Example

Consider the following picture of the graph of $f(x) = 2x+1$ on the interval [0,3).
The minimum value is 1 at
$x = 0$. But is there a maximum?

No. Because the function is
defined on the interval [0,3) which excludes the point x = 3.
So note that you can get very close to 7 as the maximum value as you get very close to x = 3, but this is in a limiting process, and you can always get more closer to 7, and yet never EQUAL 7! So 7 looks like a max, but its NOT!



The question of interest given a function f(x) is:
Does f (x) has a maximum (minimum) value?
If f (x) has a maximum (minimum) , what is it?
If f (x) has a maximum value, where does it occur?

THEOREM 4.6.4
(Extreme-Value Theorem)

If a function f is continuous on a closed interval [a,b], then f has both a maximum value and a minimum value on [a,b]

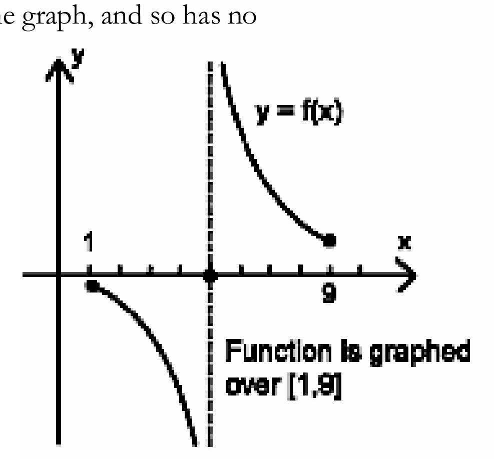
Won't prove this as its difficult. Just use it. This theorem doesn't tell us what the max and the min are, just the conditions on a function which will make it have a max or min

Example

Note that in the previous example we saw a function f (x) = 2x+1 which was defined over the interval [0, 3). This one is a continuous function on that interval, but had no maximum because the interval was not closed!

Example

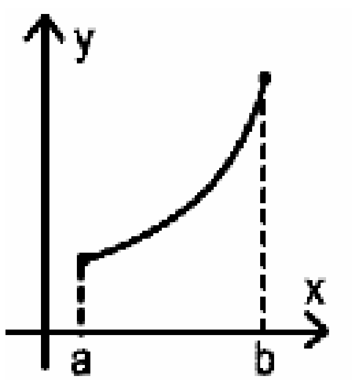
Here is another function graphed over the interval [1,9]. Although the interval is closed, the function not continuous on this interval as we can see from the graph, and so has no maximum or minimum values on that interval .

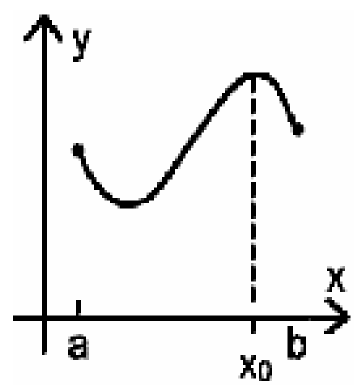


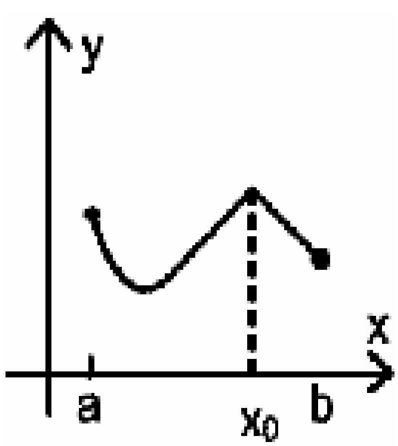
THEOREM 4.6.5

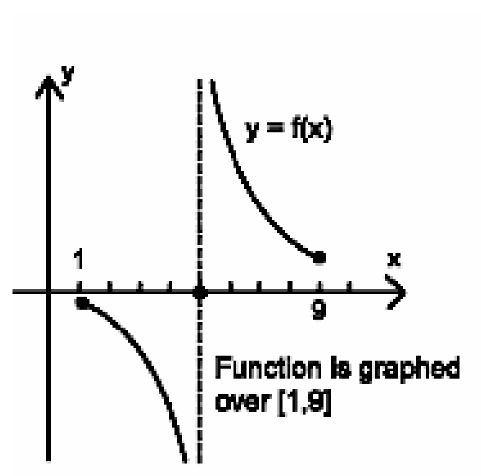
If a function f has an extreme value (either a maximum or a minimum) on an open interval(a,b), then the extreme value occurs at a critical point of f.

Step 1: Find the critical points of *f* in (a,b)
Step 2: Evaluate *f* at all the critical points and the endpoint a and b
Step 3: The largest of the values in Step 2 is the maximum value of *f* on [a,b] and the smallest is the minimum.









Example

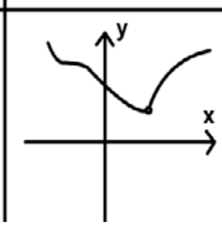
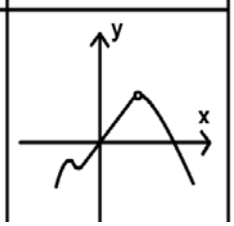
Find the maximum and the minimum values of $f(x) = 2x^3 - 15x^2 + 36x$ on the interval [1, 5]
Since *f* is a polynomial, it's continuous and differentiable on the interval (1, 5)

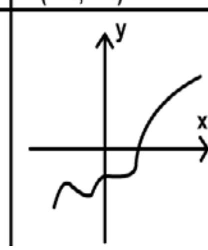
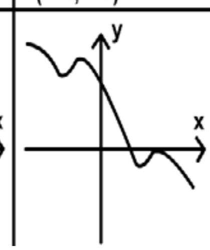
$$f^{/}(x) = 6x^2 - 30x + 36 = 0$$
$$f^{/}(x) = (x-3)(x-2) = 0$$

So $f^{/}$ is 0 at x = 2 and x = 3. Max or min will occur at these two points or at the end points.
Evaluate the function at the critical points and the endpoints and we see that max is 55 at x = 5 and min is 23 at x = 1

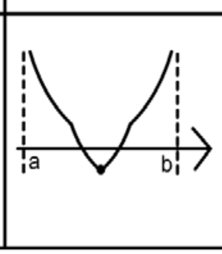
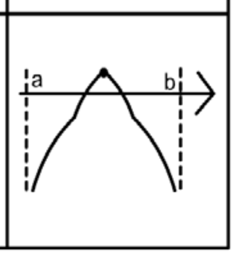
We want to know the max and min over $(-\infty, +\infty)$

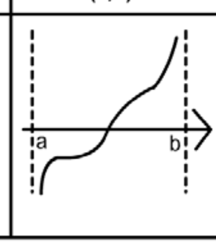
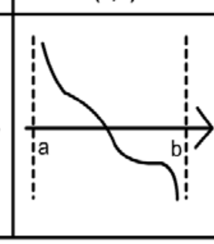
Here is how to find them for continuous functions

Limits	$\lim_{x\to-\infty} f(x) = +\infty$ $\lim_{x\to+\infty} f(x) = +\infty$	$\lim_{x\to-\infty} f(x) = -\infty$ $\lim_{x\to+\infty} f(x) = -\infty$
Conclusion if f is continuous	f has a minimum but no maximum on $(-\infty,+\infty)$	f has a maximum but no minimum on $(-\infty,+\infty)$
Graph		

Limits	$\lim_{x\to-\infty} f(x) = -\infty$ $\lim_{x\to+\infty} f(x) = +\infty$	$\lim_{x\to-\infty} f(x) = +\infty$ $\lim_{x\to+\infty} f(x) = -\infty$
Conclusion if f is continuous	f has neither a maximum nor a minimum on $(-\infty,+\infty)$	f has neither a maximum nor a minimum on $(-\infty,+\infty)$
Graph		

Summary of extreme behaviors of functions over (a,b)

Limits	$\lim_{x\to a^+} f(x) = +\infty$ $\lim_{x\to b^-} f(x) = +\infty$	$\lim_{x\to a^+} f(x) = -\infty$ $\lim_{x\to b^-} f(x) = -\infty$
Conclusion if f is continuous on (a,b)	f has a minimum but no maximum on (a,b)	f has a maximum but no minimum on (a,b)
Graph		

Limits	$\lim_{x\to a^+} f(x) = -\infty$ $\lim_{x\to b^-} f(x) = +\infty$	$\lim_{x\to a^+} f(x) = +\infty$ $\lim_{x\to b^-} f(x) = -\infty$
Conclusion if f is continuous on (a,b)	f has neither a maximum nor a minimum on (a,b)	f has neither a maximum nor a minimum on (a,b)
Graph		

Find the max and min values if any, of the function $f(x) = x^4 + 2x^3 - 1$ on the interval $(-\infty, +\infty)$

This is a continuous function on the given interval and

$$\lim_{x\to+\infty}(x^4 + 2x^3 - 1) = +\infty \quad \text{and} \quad \lim_{x\to-\infty}(x^4 + 2x^3 - 1) = +\infty$$

So *f* has a minimum but no maximum on (-inf, +inf). By Theorem 4.6.5, the min must occur at a critical point. So

$$f'(x) = 4x^3 + 6x^2 = 2x^2(2x+3) = 0$$

This gives x = 0, and x = -3/2 as the critical points. Evaluating gives min = -43/16 at x = -3/2

Applied maximum and minimum problems

We will use what we have learnt so far to do some applied problems in OPTIMIZATION. Optimization is the way efficiency is got in business, machines and even in nature in terms of animals competing for resources.

Problems involving continuous functions and Finite closed intervals

These are problems where the function is defined over a closed interval. These problems always have a solution because it is guaranteed by the extreme values theorem.

Example

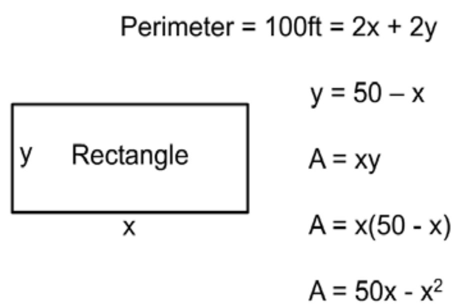
Find the dimensions of a rectangle with perimeter 100 ft whose area is as large as possible.

Let

x = length of the rectangle in feet

y = width of the rectangle in feet

A = area of the rectangle



Then A = xy

We want to maximize the area

$$A = x(50-x) = 50x - x^2$$

Perimeter = 100ft = 2x + 2y or y = 5 – x

Use this values of y in the equation A = xy to get A function of *x*

Because *x* represents a length, it cannot be negative and it cannot be a value that exceeds the perimeter of 100 ft. So we have the following constraints on *x*

$$0 \le x \le 50$$

So the question is of finding the max of $A = 50x - x^2$ on the interval [0,50]. By what we have seen so far, that max must occur at the end points of this interval or at a critical point

Critical points:

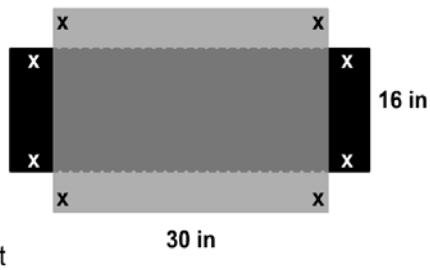
$$\frac{dA}{dx} = 50 - 2x = 0 \quad \Rightarrow \quad x = 25$$

So now we substitute x = 0, x = 25, and x = 50 into the function A to get the max 625 at the point x = 25.

Note that y = 25 also for x = 25. So the rectangle with perimeter 100 with the greatest area is a square with sides 25ft.

Example

An open box is to be made from a 16 inch by 30 inch piece of cardboard by cutting out squares of equal size from the 4 corners and bending up the sides. What size should the squares be to obtain a box with largest possible volume?



Let

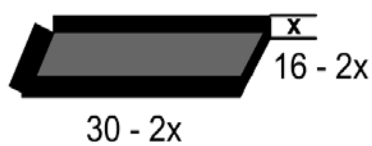
x = length of the sides of the squares to be cut out

V = volume of the resulting box.

We want to maximize the Volume V.

If we cut out the squares from the 4 corners of the cardboard, the resulting BOX will have dimensions

$(16 – 2x)$ by $(30 – 2x)$



$$\begin{aligned} V &= (\text{length})(\text{width})(\text{Height}) \\ &= (16-2x)(30-2x)x \\ &= 480x - 92x^2 + 4x^3 \end{aligned}$$

$$0 \le x \le 8$$

s

So want to find max of the function on [0,8].

$$\frac{dV}{dx} = 480 - 184x + 12x^2 = 0 \quad \Rightarrow \quad x = \frac{10}{3} \text{ and } x = 12$$

Because x = 12 is out of [0, 8], ignore it. Check V at the end points and at x = 10/3. We see then that V = 19600/27 is max when x = 10/3.