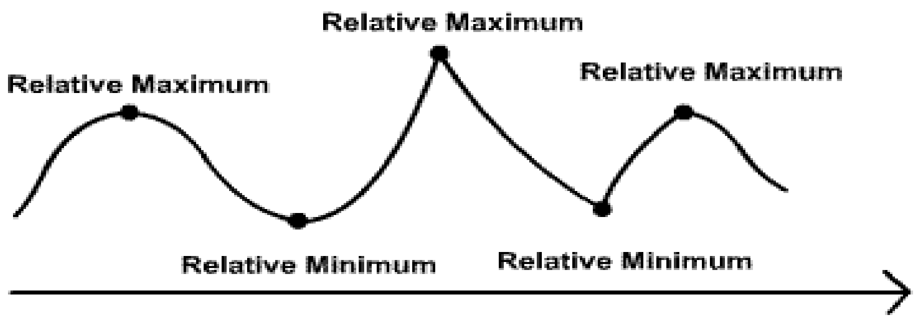
Lecture # 22

Relative Extrema

- Relative Maxima
- Relative Minima
- Critical Points
- First Derivative test
- Second Derivative test
- Graphs of Polynomials
- Graphs of Rational functions

Relative Maxima



Most of the graphs we have seen have ups and downs, much like Hills and valleys on earth.
The Ups or the Hills are called relative Maxima.
The downs or the Valleys are called relative Minima
The reason we use the word relative is that just like a given Hill in a mountain range need not necessarily be the Highest point in the range. Similarly a given maxima in a graph need not be the maximum possible value in the graph.
Same goes for the relative minima.
In general, we may say that a given Hill is the highest one in some area. Look at relative maxima in a given interval.
Again, same is true for valleys and relative minima.
So when we talk about relative maxima and relative minima, we talk about them in the context of some interval.

Definition 4.3.1

A function f is said to have a relative maximum at x_0 if $f(x_0) \geq f(x)$ for all x in some open interval containing x_0

Definition 4.3.2

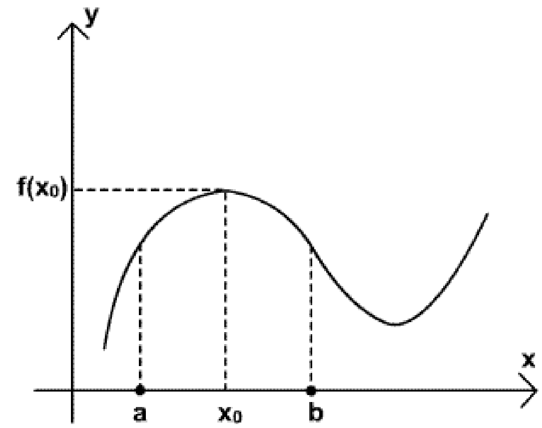
A function is said to have a relative minimum at x_0 if $f(x_0) \leq f(x)$ for all x in some open interval containing x_0.

Definition 4.3.3

A function f is said to have a relative extremum at x_0 if it has either a relative maximum or relative minimum at x_0

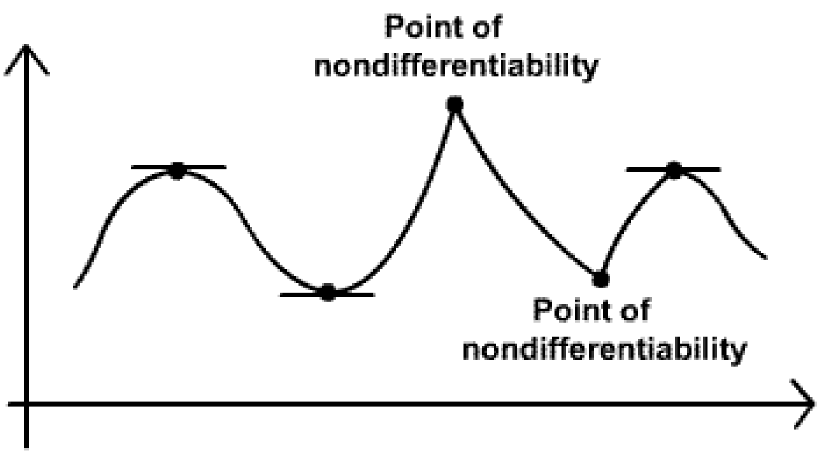
Example
Here is a graph of a function *f*. This has a relative maximum in the interval (a, b) because from the graph its obvious that

$f(x_0) \geq f(x)$



Function has a relative maximum in the interval (a,b)

Critical Points
It so happens that relative extrema can be viewed as transition points that separate the regions where a graph of a function is increasing from those where a graph is decreasing.



Here is a figure. This shows that relative extrema of a function occur at points where *f* has a horizontal tangent, or where the function is not differentiable.
Horizontal tangent means derivative = 0.
Non-differentiable means corners.

Theorem 4.3.4

If f has a relative extremum at x_0, then either $f'(x_0) = 0$ or f is not differentiable at x_0

Definition 4.3.5

A critical point for a function f is any value of x in the domain of f at which $f'(x) = 0$ or at which f is not differentiable; the critical points where $f'(x) = 0$ are called stationary points of f.

So theorem 4.3.4 can be read as now with this new terminology As "The relative extrema of a function, if any, occur at critical points."

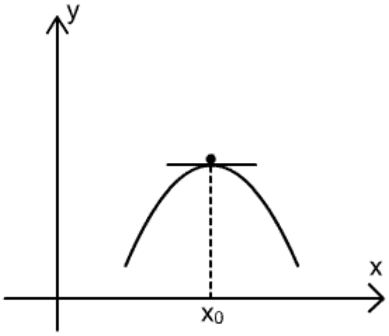
Example

a) x_0 here is a critical and stationary point as tangent line has slope 0

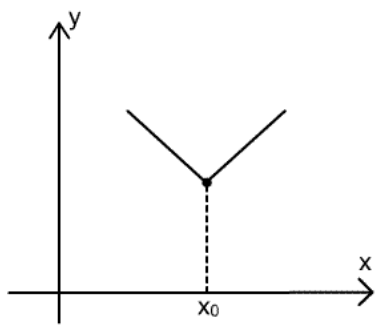
f) x_0 here is a critical point and it has minimum value at that point but the tangent line is not defined at that point.

g) x_0 is a critical point but not stationary as derivative does not exist

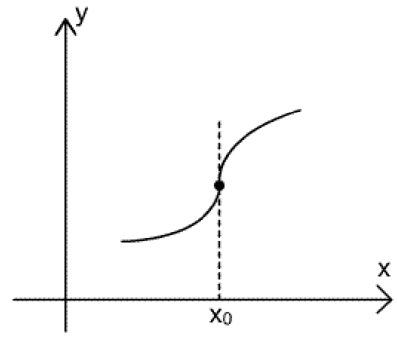
Here are the figures of the situations.



4.3.4 (a)



4.3.4 (f)



4.3.4 (g)

First Derivative test and Second Derivative test

Note that in (g) of the last figure, x_o was a critical point, but there was not relative extrema there! This can happen.
So how do we know at which critical point a relative extrema occurs or not?
Here is a theorem for that.

THEOREM 4.3.6
(First Derivative Test)

a) If $f'(x) > 0$ on an open interval extending left from x_0 and $f'(x) < 0$ on an open interval extending right from x_0, then f has a relative maximum at x_0 from x_0 and $f'(x) > 0$ on an open interval

b) If $f'(x) < 0$ on an open interval extending left
extending right from x_0, then f has a rel. minimum at x_0

(c) If $f'(x)$ has the same sign [either $f'(x) > 0$ or $f'(x) < 0$] on an open interval extending left from x_0 and on an open interval extending right from x_0, then f does not have a relative extremum at x_0

In short: " The relative extrema, if any, on an open interval where a function f is continuous and not constant occurs at those critical points where f' changes sign"

Example

Locate the relative extrema of

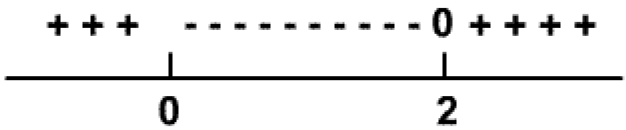
$$f(x) = 3x^{\frac{5}{3}} - 15x^{\frac{2}{3}}$$

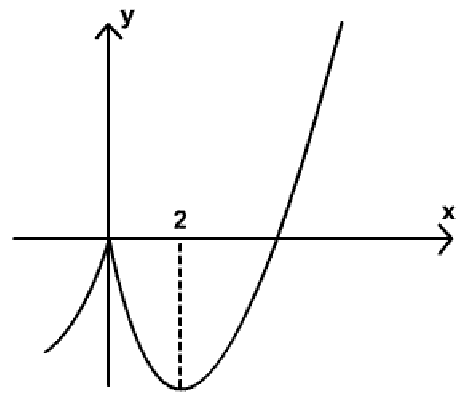
$$f'(x) = 5x^{\frac{2}{3}} - 10x^{-\frac{1}{3}} = 5x^{-\frac{1}{3}}(x-2) = \frac{5}{x^{\frac{1}{3}}}(x-2)$$

Note that there are two critical points, namely x = 0 and x = 2 Because at x = 2, the derivative $f^{/}$ = 0, and at x = 0, the derivative does not exists.

Now we need to know where there is a relative extrema by checking for the changing sign of $f^{/}$ at the 2 critical points.

We use theorem 4.3.6 and draw a number line test





We see that there is relative maximum at 0
We see that there is relative minimum at 2

There is another test for finding extrema easier than the first derivative test.

Theorem 4.3.7 (Second Derivative Test)

Suppose that f is twice differentiable at a stationary point x_0

(a) If $f''(x_0) > 0$, then f has relative minimum at x_0.

(b) If $f''(x_0) < 0$, then f has relative maximum at x_0.

EXAMPLE

Locate the relative extrema of $f(x) = x^4 - 2x^2$

$$f^{/}(x) = 4x^3 - 4x = 4x(x-1)(x+1)$$

$$f^{//}(x) = 12x^2 - 4$$

Setting $f^{/}(x) = 0$ gives stationary points $x = 0$ and $x = \pm 1$

Also,

$$f^{//}(0) = -4 < 0$$

$$f^{//}(1) = 8 > 0$$

$$f^{//}(-1) = 8 > 0$$

There is a relative maximum at x = 0, and relative minima at x = 1 and x = -1

Graphs of Polynomials

In applied sciences and engineering, it is required many times to understand the behavior of a function.
Graphs are a good way to understand function behavior. But many times it is hard to graph the function.
So it is often necessary to understand the behavior in terms of maxima and minima and concavity etc.
We will look at how the stuff from the last lecture helps us in graphing polynomial and rational functions
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Graphs of Polynomials

Let P(x) be a polynomial function

- Calculate $P'(x)$ and $P''(x)$
- Using P', determine the stationary points and the intervals of increase and decrease
- Use P'' to determine the inflection points and interval where P is concave up and concave down
- Plot all of the above and the x and y intercepts

Example

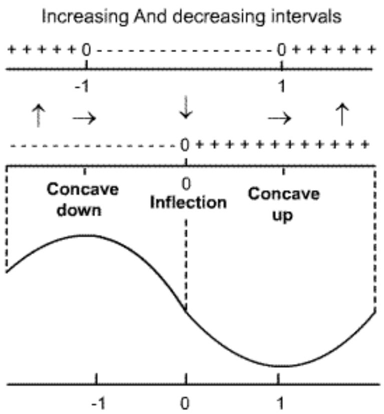
Sketch the graph of $P(x) = y = x^3 - 3x + 2$

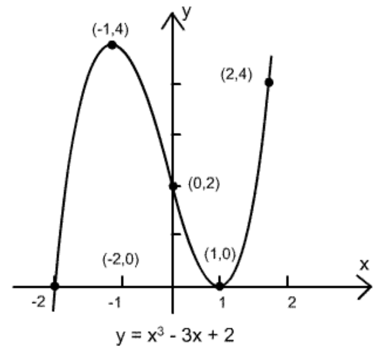
$$\frac{dy}{dx} = 3x^2 - 3 = 3(x-1)(x+1)$$

$$\frac{d^2y}{dx^2} = 6x$$

You find the stationary points, inflection points.

x	$y = x^3 - 3x + 2$
−2	0
−1	4
0	2
1	0
2	4





Above figure Shows the intervals of increase/decrease and of concavity Y intercept at (0,2) Inflection point is at x = 0

Graph of Rational Functions

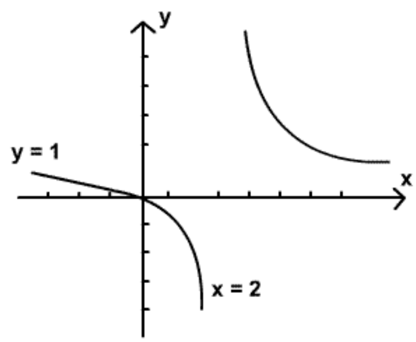
Rational function, remember, is a function defined by the ratio of two polynomials

$$R(x) = \frac{P(x)}{Q(x)}$$

Its obvious that if Q(x) = 0, then R(x) has discontinuity at those values of *x* where Q(x) = 0

Consider the following graph of

$$f(x) = \frac{x}{x-2}$$



A line $x = x_0$ is called a vertical asymptots for the graph of a function f if $f(x) \to +\infty$ or $f(x) \to -\infty$ as x approaches x_0 from the right or from the left .A line $y = y_0$ is called a horizental asymptote for the graph of f if

$$\lim_{x \to +\infty} f(x) = y_0 \quad \text{or} \quad \lim_{x \to -\infty} f(x) = y_0$$

Vertical asymptotes occur where the denominator is 0

Example
Find horizontal and vertical asymptotes of

$$f(x) = \frac{x^2 + 2x}{x^2 - 1}$$

The vertical asymptotes occur at the points where $x^2 - 1 = 0$; these are the points $x = -1$ and $x = 1$, Since

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{x^2 + 2x}{x^2 - 1} = \lim_{x \to +\infty} \frac{x^2}{x^2} = 1$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{x^2 + 2x}{x^2 - 1} = \lim_{x \to -\infty} \frac{x^2}{x^2} = 1$$

It follows that $y = 1$ is a horizental asymptote.