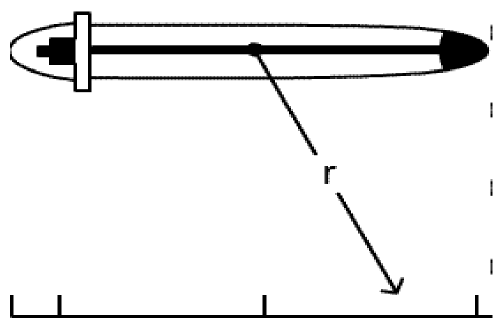
Lecture # 21

Applications of Differentiation

- Related Rates
- Increasing Functions
- Decreasing Functions
- Concavity of functions

Related Rates
Related Rates are real life problems.
These involve finding the rate at which one quantity changes w.r.t another quantity.
For example, we may be interested in finding out how fast the polar ice caps are melting w.r.t the changes in temperature.
We may want to know how fast a satellite is changing altitude w.r.t changes in time, or w.r.t changes in gravity.
To solve problems involving related rates, we use the idea of derivatives, which measure the rate of change.

Example
Assume that oil spilled from a ruptured tanker spread in a circular pattern whose radius increases at a constant rate of 2 ft/sec. How fast is the area of the spill increasing when the radius of the spill is 60ft?
Let t = number of seconds elapsed from the time of the spill
Let r = radius of the spill in feet after t seconds
Let A = area of the spill in square feet after t seconds



We want to find $\left.\dfrac{dA}{dt}\right|_{r=60}$ given that $\dfrac{dr}{dt} = 2\, ft/\sec$

The spill is circular in shape so $A = \pi r^2$

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt} \quad \Rightarrow \quad \left.\frac{dA}{dt}\right|_{r=60} = 2\pi(60)(2) = 240\pi\, ft^2/\sec$$

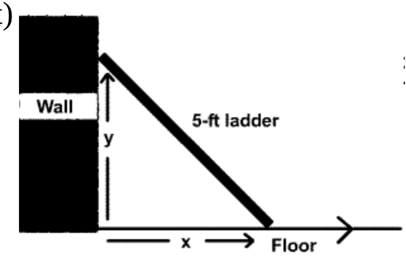
The following steps are helpful in solving related rate problems
1. Draw a figure and label the quantities that change
2. Identify the rates of change that are known and those that are to be found
3. Find an equation that relates the quantity whose rate of change is to be found to those quantities whose rates of change are known.
4. Differentiate the equation w.r.t the variable that quantities are changing in respect to. Usually Time.
5. Evaluate the derivative at appropriate points.

Example

A five foot ladder is leaning against a wall. It slips in such a way that its base is moving away from the wall at a rate of 2 ft/sec at the instant when the base is 4ft from the wall. How fast is the top of the ladder moving down the wall at that instant?
Let t = number of seconds after the ladder starts to slip
Let x = distance in feet from the base of the ladder
Let y = distance in feet from the top of the ladder to the floor.

$\frac{dx}{dt} =$ rate of change of the base of the ladder (horizontal movement)

$\frac{dy}{dt} =$ rate of change of the top of the ladder (vertical movement)



Using Pythagoras theorem
$x^2 + y^2 = 5^2 = 25$

We want $\left.\frac{dy}{dt}\right|_{x=4}$ Given That $\left.\frac{dx}{dt}\right|_{x=4} = 2\, ft/\sec$
How do we relate the x and the y?
Look at the picture again I see Pythagoras's theorem in here!

$$x^2 + y^2 = 25$$

Differentiating wrt t and using chain rule gives

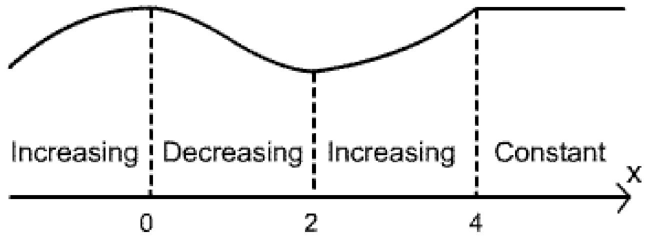
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

$$\left.\frac{dy}{dt}\right|_{x=4} = -\frac{4}{3}(2) = -\frac{8}{3}\, ft/\sec$$

When x = 4, use the above equation to find corresponding y.

Increasing and decreasing functions
We saw earlier in the lectures that we can get an idea of the graph of a function by plotting a few values. But remember that we also said that this graph was an approximation as a few points may not give all the info.
Now we will see that we can use derivatives to get accurate info about the behavior of the graph in an interval when we move from left to right.
Increasing function on an interval means that as we move from left to right in the x-direction, the y-values increase in magnitude.
Decreasing function on an interval means that as we move from left to right in the x-direction, the y-values decrease in magnitude.
An interval means that as we move from left to right in the x-direction, the y-values decrease in magnitude.
The graph of the function in this figure shows that the function is increasing on the intervals and $(-\infty, 0)$ and $[2, 4]$ decreasing on the interval [0,2].



Let's make this idea concrete.

Definition 4.2.1

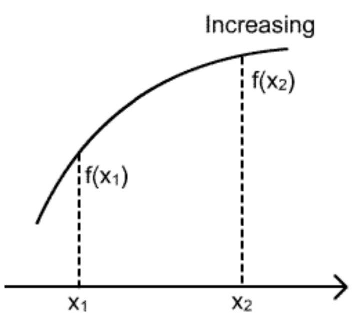
Let f be defined on an interval, and let x_1 and x_2 denote points in that interval.

(a) f is increasing on the interval if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$

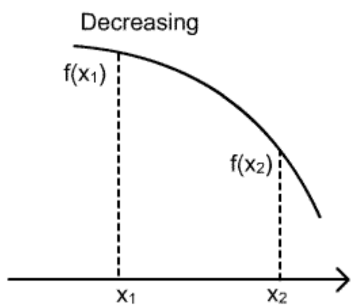
(b) f is decreasing on the interval if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$

(c) f is constant on the interval if $f(x_1) = f(x_2)$ for all x_1 and x_2

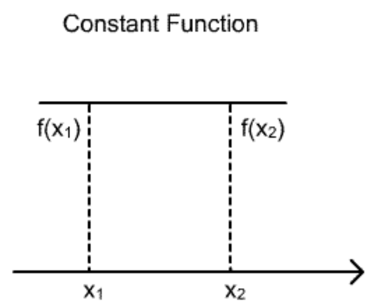
As shown in the figures below



$f(x_1) < f(x_2)$ if $x_1 < x_2$

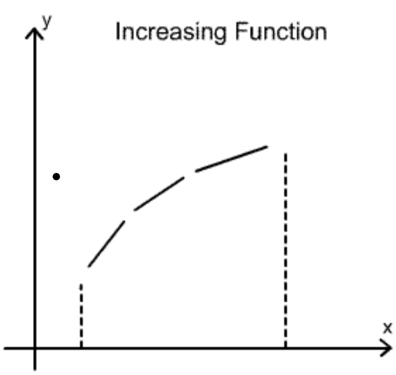


$f(x_1) > f(x_2)$ if $x_1 < x_2$

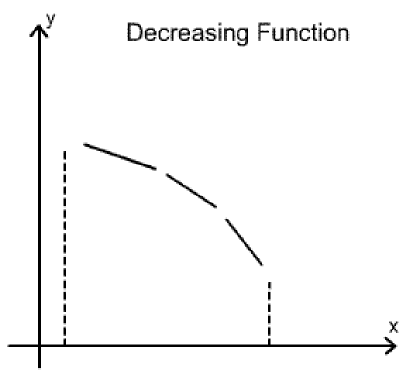


$f(x_1) = f(x_2)$ for all x_1 , x_2

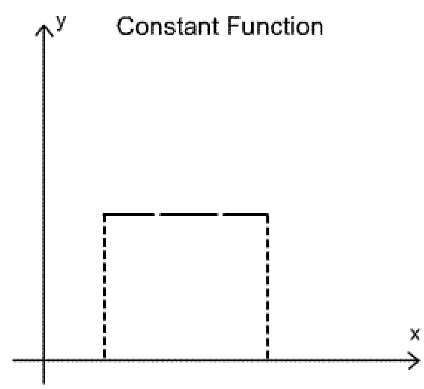
Let's take a few points on the 3 graphs in above figures and make tangent lines on these points. This gives



Tangent lines have positive slope



Tangent lines have negative slope



Tangent lines have zero slope

Note that incase where the graph was increasing, we get tangent line with positive slopes, decreasing we get negative slope, and constant gives 0 slope.

Let's formalize this idea

Theorem 4.2.1

Let f be a function that is continuous on a closed interval [a,b] and differentiable on the open interval (a,b)

(a) If $f'(x) > 0$ for every value of x in (a,b), then f is increasing on [a,b].

(b) If $f'(x) < 0$ for every value of x in (a,b), then f is decreasing on [a,b].

(c) If $f'(x) = 0$ for every value of x in (a,b), then f is constant on [a,b].

Example

Find the intervals on which the function is increasing and those on which its decreasing.

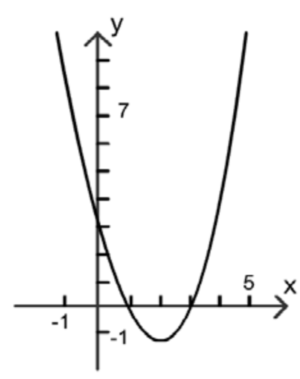
$$f(x) = x^2 - 4x + 3$$

Differentiating f gives

$$f'(x) = 2x - 4 = 2(x-2)$$

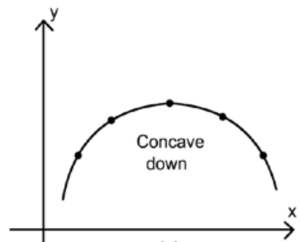
$$f'(x) < 0 \;\Rightarrow\; 2(x-2) < 0 \;\Rightarrow\; -\infty < x < 2$$

$$f'(x) > 0 \;\Rightarrow\; 2(x-2) > 0 \;\Rightarrow\; 2 < x < +\infty$$

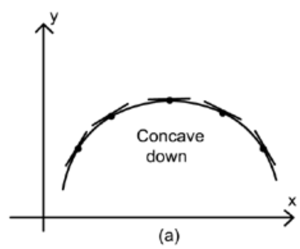


$f(x) = x^2 - 4x + 3$

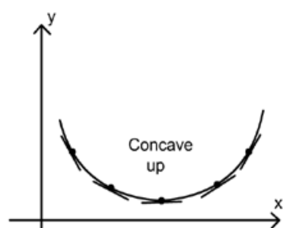
Since f is continuous on (2,+∞), the function is actually increasing on the interval [2,+∞)
Similarly it is decreasing on the interval (-∞,2]
The derivative is 0 at the point x=2
Since this is the only point not the interval we think of the point x=2 as the point where the transition occurs from decreasing to increasing in f.



(a)



(a)
Slope of tangent lines decreasing



(b)
Slope of tangent lines increasing

Definition 4.2.3

Let f be differentiable on an interval

(a) f is called concave up on the interval if f is increasing on the interval.

(b) f is called concave down on the interval if f is decreasing on the interval.

Example

Find the open intervals on which the given function is concave up and on those on which it is concave down.

$$f(x) = x^2 - 4x + 3$$

$$f'(x) = 2x - 4$$

$$f''(x) = 2$$

THEOREM 4.2.2

(a) If $f''(x) > 0$ on a open interval (a,b) then f is concave up on (a,b)

(b) If $f''(x) < 0$ on a open interval (a,b) then f is concave down on (a,b)

$$f''(x) = 2$$

Since f'' is greater than 0 for all x , the graph of this function is concave up on the interval $(-\infty \ , +\infty)$