Lecture # 20

Derivatives of Logarithmic and Exponential Functions and Inverse functions and their derivatives

- Derivative of the logarithmic function $f(x) = \log_b(x)$
- Derivative of the Natural log functions $f(x) = \ln(x)$
- Logarithmic Differentiation
- Derivatives of Irrational powers of *x*
- Derivatives of Exponential functions $f(x) = a^x$
- Inverse functions
- Derivatives of Inverse Functions

Derivative of the logarithmic function

$y = f(x) = \log_b(x)$

Recall the Logarithm of a real number. Let *x, b* be real, and an unknown real number *y.*
What should *y* be so that if you raise *y* to *b,* you get *x*.

That is $b^y = x$

This is denoted as $y = f(x) = \log_b(x)$

b is called the *base.*

The equation above is read as "log *base b* of *x* is *y* .We want to find the derivative of the log function. Before we do that, here are some formulas we will need that follow from the properties of log functions. You can refer to them in section 7.1 of your text.

Theorem 7.1.1

(a) $\log_b 1 = 0$

(b) $\log_b b = 1$

(c) $\log_b ac = \log_b a + \log_b c$

(d) $\log_b \frac{a}{c} = \log_b a - \log_b c$

(e) $\log_b a^r = r \log_b a$

(f) $\log_b \frac{1}{c} = -\log_b c$

Let's apply the derivative on both sides and use its definition to get

$$e = \lim_{x\to 0}(1+x)^{\frac{1}{x}} \qquad \therefore\ e \approx 2.71$$

$$\frac{dy}{dx} = \frac{d}{dx}[\log_b(x)] = \lim_{h\to 0}\frac{\log_b(x+h) - \log_b(x)}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\log_b\left(\frac{x+h}{x}\right)$$

$$= \lim_{h\to 0}\frac{1}{h}\log_b\left(1+\frac{h}{x}\right)$$

$$= \lim_{v\to 0}\frac{1}{vx}\log_b(1+v)$$

Last step comes from letting

$v = \frac{h}{x}$ and then $v \to 0$ as $h \to 0$

$$= \frac{1}{x} \lim_{v \to 0} \frac{1}{v} \log_b (1+v)$$

$$= \frac{1}{x} \lim_{v \to 0} \log_b (1+v)^{\frac{1}{v}}$$

Using Theorem 7.1.1 part (e)

$$= \frac{1}{x} \log_b \left[\lim_{v \to 0} (1+v)^{\frac{1}{v}} \right]$$

Using the fact that log is continuous

$$= \frac{1}{x} \log_b e$$

Thus

$$\frac{d}{dx}[\log_b(x)] = \frac{1}{x} \log_b e, \; x > 0$$

There exists a "change of base" formula which Let's you convert your given log to a certain base to another. This is helpful because modern calculators have two log button, one with base 10, written as only log (x) and the other for the Natural log to the base *e* written as ln (x). To keep calculations easy, we can rewrite the above formula for the derivative as

$$\frac{d}{dx}[\log_b(x)] = \frac{1}{x \ln(b)}, \; x > 0$$

If the base happens to be the number *e,* then we get from this formula

$$\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x}, \; x > 0$$

So you see that the the simplest formula for the derivative comes from base $b = e$. That's why ln (x) is a very important function in terms of Calculus

Derivative of the Natural log functions

$f(x) = \ln(x)$

We will generalize the derivative formula we just found for the log or ln function to include composition of functions where one is the ln function, and the other is some function u(x) > 0 and differentiable at x.

$$\frac{d}{dx}[\log_b(u)] = \frac{1}{u \ln(b)} \cdot \frac{du}{dx}$$

Using the chain rule here

$$\frac{d}{dx}[\ln(u)] = \frac{1}{u} \cdot \frac{du}{dx}$$

Example

Find $\frac{d}{dx}\left[\ln(x^2+1)\right]$

$$\frac{d}{dx}\left[\ln(x^2+1)\right] = \frac{1}{(x^2+1)}\cdot\frac{d}{dx}[x^2+1]$$

$$= \frac{1}{(x^2+1)}\cdot 2x = \frac{2x}{(x^2+1)}$$

Example

$$\frac{d}{dx}\left[\ln\left(\frac{x^2\sin(x)}{\sqrt{1+x}}\right)\right]$$

$$= \frac{d}{dx}\left[\ln[x^2\sin(x)] - \ln\sqrt{1+x}\right] = \frac{d}{dx}\left[2\ln(x) + \ln(\sin x) - \frac{1}{2}\ln(1+x)\right]$$

$$= \frac{2}{x} + \frac{\cos(x)}{\sin(x)} - \frac{1}{2(1+x)}$$

Logarithmic Differentiation Using log function properties, we can simplify differentiation of messy functions.

Example

$$y = \frac{x^2 \,.\, \sqrt[3]{7x-14}}{(1+x^2)^4}$$

Apply ln to both sides and simplify the right side using ln properties

$$\ln y = \ln\left(\frac{x^2 \,.\, \sqrt[3]{7x-14}}{(1+x^2)^4}\right) = 2\ln x + \frac{1}{3}\ln(7x-14) - 4\ln(1+x^2)$$

Differentiating both sides w. r. t. x gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{\frac{7}{3}}{7x-14} - \frac{8x}{1+x^2}$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{2}{x} + \frac{\frac{7}{3}}{7x-14} - \frac{8x}{1+x^2}\right)\cdot y$$

$$\frac{dy}{dx} = \left(\frac{2}{x} + \frac{\frac{7}{3}}{7x-14} - \frac{8x}{1+x^2}\right)\cdot\frac{x^2 \,.\, \sqrt[3]{7x-14}}{(1+x^2)^4}$$

Derivatives of Irrational powers of *x*

Remember that we proved the power rule for positive integers, then for all integers, then for rational numbers?

Now we prove it for all REAL numbers by proving it for the remaining type of real numbers, namely the irrationals.

Power Rule

$$\frac{d}{dx}\left[x^r\right] = r \cdot x^{r-1}$$

We will use ln to prove that the power rule holds for any real number r

Let $y = x^r$, r any real number

Proceed by differentiating this function using ln

$$\ln y = \ln x^r = r \ln x$$

$$\frac{d}{dx}[\ln y] = \frac{d}{dx}[r \ln x]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{r}{x}$$

$$\frac{dy}{dx} = \frac{r}{x} \cdot y = \frac{r}{x} \cdot x^r = r \cdot x^{r-1}$$

Derivatives of Exponential functions $f(x) = b^x$

Now we want to find the derivative of the function $y = f(x) = b^x$

To do so, we will use logarithmic differentiation on $y = b^x$

$$\ln y = \ln b^x = x \ln b$$

$$\frac{d}{dx}[\ln y] = \frac{d}{dx}[x \ln b]$$

$$\frac{1}{y}\frac{dy}{dx} = \ln b$$

$$\frac{dy}{dx} = y \cdot \ln b = b^x \cdot \ln b$$

$$So \quad \frac{d}{dx} b^x = b^x \cdot \ln b$$

In general

$$\frac{d}{dx}[b^u] = b^u \cdot \ln b \cdot \frac{du}{dx}$$

If $b = e$

$$\frac{d}{dx} e^x = e^x \cdot \ln e = e^x$$

In general

$$\frac{d}{dx}[e^u] = e^u \cdot \ln e \cdot \frac{du}{dx} = e^u \cdot \frac{du}{dx}$$

Inverse Functions

We have talked about functions.

One way of thinking about functions was to think of a function as a process that does something to an input and then throws out an output.
So this way, when we put in a real number *x*, it comes out as some new number *y,* after some ACTION has taken place.
Now that question is, that is there another ACTION, that undoes the first action?
If *f* is a function that performs a certain action on *x,* is there a function *g* that undoes what *f* does?
Sometimes there is and sometime not. When there is a function *g* undoes what *f* does, then we say that *g* is an inverse function of *f.*
Here is how to determine if two given function are inverses of each other or not.

Definition 7.4.1

If the functions *f* and *g* satisfy the two conditions

f(g(x)) = *x* for every *x* in the domain of *g*
g(f(x)) = *x* for every *x* in the domain of *f*

then we say , that *f* is an inverse of *g* and *g* is an inverse of *f* or, alternatively that *f* and *g* are inverse functions.

Example

$f(x) = 2x$ and $g(x) = \frac{1}{2}x$

are inverse functions. This is obvious, but we can use def 7.4.1

$$f(g(x)) = 2(\frac{1}{2}x) = x$$

$$g(f(x)) = \frac{1}{2}(2x) = x$$

Inverses don't always exists for a given function.
There are conditions for it.
Most important one is that a function must be ONE to ONE to have an inverse.
Do you recall what One to One is??
Its when a function does not have two values for the same *x* value in the domain. Review this.
The inverse function to a function *f* is usually denoted as f^{-1} read as "*f* inverse" and

$$y = f^{-1}(x) \quad \Rightarrow \quad x = f(y)$$

Derivatives of Inverse Functions
Theorem 7.4.7

Suppose that the function f has an inverse and that the value of $f^{-1}(x)$ varies over an interval on which f has a nonzero derivative as x varies over an interval I. Then f^{-1} is differentaible on I and derivative of f^{-1} is given by the formula

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

The formula in this theorem can be simplified if you write

$$y = f^{-1}(x) \quad \Rightarrow \quad x = f(y)$$

Then $\dfrac{dy}{dx} = (f^{-1})'(x) \quad \Rightarrow \quad \dfrac{dx}{dy} = f'(y) = f'(f^{-1}(x))$

Plugging these values into the theorem we get a simpler formula

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$