Lecture 2

Absolute Value

In this lecture we shall discuss the notation of Absolute Value. This concept plays an important role in algebraic computations involving radicals and in determining the distance between points on a coordinate line.

Definition

The absolute vale or magnitude of a real number **a** is denoted by **|a|** and is defined by

$$|a| = \begin{cases} a & if\ a \ge 0, that\ is, a\ is\ non-negative \\ -a & if\ a < 0, that\ is, a\ is\ negative. \end{cases}$$

Technically, 0 is considered neither positive, nor negative in Mathematics. It is called a non-negative number. Hence whenever we want to talk about a real number a such that **a** $\geq$ 0, we call **a** non-negative, and positive if **a** > 0.

Example

$|5| = 5$, $\left|-\frac{4}{7}\right| = -(-\frac{4}{7}) = \frac{4}{7}$, $|0| = 0$

since 5>0 since -4/7 < 0 since 0 $\geq$ 0

Note that the effect of taking the absolute value of a number is to strip away the minus sign if the number is negative and to leave the number unchanged if it is non-negative. Thus, |a| is a non-negative number for all values of a and $-|a| \le a \le |a|$, if 'a' is itself is negative, then '–a' is positive and '+a' is negative.

$$a + b \ge 0\ or\ a + b < 0$$

$$a + b = |\ a + b\ |$$

$$|a + b| \le |a| + |b|$$

$$|a + b| = -(a + b)$$

Caution: Symbols such as +a and –a are deceptive, since it is tempting to conclude that +a is positive and –a is negative. However this need not be so, since a itself can represent either a positive or negative number. In fact , if a itself is negative, then –a is positive and +a is negative.

Example: Solve $|x-3|=4$

Solution:

Depending on whether x-3 is positive or negative , the equation |x-3| = 4 can be written as

x-3 = 4 or x-3 = -4

Solving these two equations give

x=7 and x=-1

Example Solve $|3x-2|=|5x+4|$

Because two numbers with the same absolute value are either equal or differ only in sign, the given equation will be satisfied if either

$$\begin{array}{lll} 3x-2=5x+4 & or & 3x-2=-(5x+4) \\ 3x-5x=4+2 & or & 3x-2=-5x-4 \\ -2x=6 & or & 3x+5x=-4+2 \\ x=-3 & or & x=-\dfrac{1}{4} \end{array}$$

Relationship between Square Roots and Absolute Values :

Recall that a number whose square is **a** is called a square root of **a**.

In algebra it is learned that every positive real number **a** has two real square roots, one positive and one negative. The positive square root is denoted by $\sqrt{a}$ For example, the number 9 has two square roots, -3 and 3. Since 3 is the positive square root, we have $\sqrt{9}=3$.

In addition, we define $\sqrt{0}=0$.

It is common error to write $\sqrt{a^2} = a$. Although this equality is correct when a is nonnegative, it is false for negative a. For example, if a=-4, then $\sqrt{a^2} = \sqrt{(-4)^2} = \sqrt{16} = 4 \neq a$

The positive square root of the square of a number is equal to that number.

A result that is correct for all a is given in the following theorem.

Theorem: For any real number **a**, $\sqrt{a^2} = |a|$

Proof :

Since $a^2 = (+a)^2 = (-a)^2$, the number +a and –a are square roots of a^2. If $a \geq 0$, then +a is nonnegative square root of a2, and if $a < 0$, then -a is nonnegative square root of a^2. Since $\sqrt{a^2}$ denotes the nonnegative square root of a^2, we have

$$\text{if} \quad \begin{array}{ll} \sqrt{a^2} = +a & \text{if } a \geq 0 \\ \sqrt{a^2} = -a & \text{if } a < 0 \end{array}$$

That is, $\sqrt{a^2} = |a|$

Properties of Absolute Value

Theorem

If **a** and **b** are real numbers, then

(a) |-a| = |a|, a number and its negative have the same absolute value.

(b) |ab| = |a| |b|, the absolute value of a product is the product of absolute values.

(c) |a/b| = |a|/|b|, the absolute value of the ratio is the ratio of the absolute values

Proof (a) :

$$|-a| = \sqrt{(-a)^2} = \sqrt{a^2} = |a|$$

Proof (b) :

$$|ab| = \sqrt{(ab)^2} = \sqrt{a^2b^2} = \sqrt{a^2}\sqrt{b^2} = |a||b|$$

This result can be extended to three or more factors. More precisely, for any n real numbers, a1,a2,a3,……an, it follows that

$$|a_1a_2\ldots.a_n| = |a_1|\ |a_2|\ \ldots\ldots.|a_n|$$

In special case where a_1, a_2,…….,a_n have the same value, a, it follows from above equation that

$$|a^n|=|a|^n$$

Example

(a) $|-4|=|4|=4$

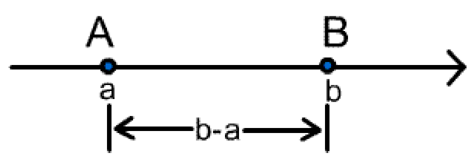
(b) $|(2)(-3)|=|-6|=6=|2|\,|-3|=(2)(3)=6$

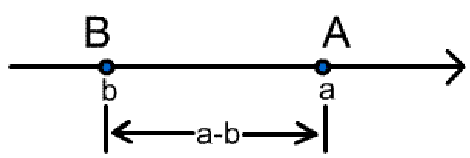
(c) $|5/4|=5/4=|5|/|4|=5/4$

Geometric Interpretation Of Absolute Value

The notation of absolute value arises naturally in distance problems, since distance is always nonnegative. On a coordinate line, let A and B be points with coordinates a and b, the distance d between A and B is

$$d=\begin{cases} b-a & \text{if } a<b \\ a-b & \text{if } a>b \\ 0 & \text{if } a=b \end{cases}$$





As shown in figure b-a is positive, so b-a=|b-a| ; in the second case b-a is negative, so

$$a-b = -(b-a) = |b-a|\ .$$

Thus, in all cases we have the following result:

Theorem

(Distance Formula)

If A and B are points on a coordinate line with coordinates a and b, respectively, then the distance d between A and B is

$$d = |b-a|$$

This formula provides a useful geometric interpretation

Of some common mathematical expressions given in table here

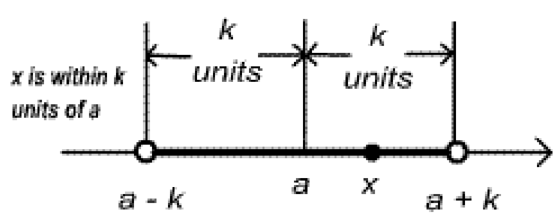
Table

EXPRESSION	GEOMETRIC INTERPRETATION ON A COORDINATE LINE
$\vert x-a\vert$	The distance between x and a
$\vert x+a\vert$	The distance between x and -a
$\vert x\vert$	The distance between x and origin

Inequalities of the form $|x-a|<k$ and $|x-a|>k$ arise often, so we have summarized the key facts about them here in following table

TABLE 1.2.2(a)

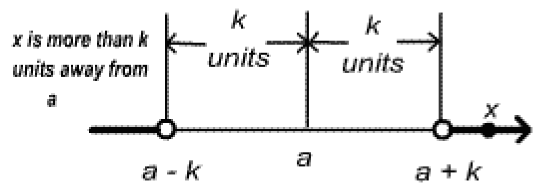
$|x - a| < k \quad (k > 0)$



Alternative Form	$-k < x - a < k$
Solution Set	$(a - k, a + k)$

TABLE 1.2.2(a)

$|x - a| > k \quad (k > 0)$



Alternative Form $\quad$ **x-a<-k or x-a>k**

Solution Set $\quad$ **(-∞,a - k)U(a + k,+∞)**

Example

Solve $|x-3| < 4$

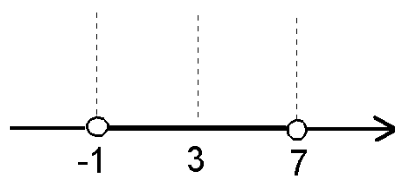
Solution: This inequality can be written as

$-4 < x-3 < 4$

adding 3 throughout we get

$-1 < x < 7$

This can be written in interval notation as (-1,7)



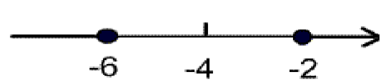
Example Solve $|x+4| \geq 2$

Solution: The given inequality can be written as

$$\begin{cases} x+4 \leq -2 \\ or \\ x+4 \geq 2 \end{cases} \qquad \text{or simply} \qquad \begin{cases} x \leq -6 \\ or \\ x \geq -2 \end{cases}$$

Which can be written in set notation as

$(-\infty, -6]U[-2, +\infty)$



(-∞, -6] U [-2, +∞)

The Triangle Inequality

$$\begin{aligned}
&\textit{It is not generally true that } |a+b| = |a|+|b| \\
&\quad \textit{For example , if } a=2 \textit{ and } b=-3 \textit{ , then } a+b=-1 \\
&\quad\quad \textit{so that } |a+b| = |-1| = 1 \\
&\textit{whereas} \\
&\quad |a| = |b| = |2| + |-3| = 2+3 = 5 \\
&\textit{so } |a+b| \neq |a|+|b|
\end{aligned}$$

It is true, however, that the absolute value of a sum is always less than or equal to the sum of the absolute values. This is the content of the following very important theorem, known as the triangle inequality . This TRIANGLE INEQUALITY is the essence of the famous HISENBURG UNCERTAINITY PRINCIPLE IN QUANTUM PHYSICS, so make sure you understand it fully.

THEOREM 1.2.5

(Triangle Inequality)

If *a* and *b* are any real numbers, then

$$|a+b| \le |a|+|b|$$

PROOF

Remember the following inequalities we saw earlier .

$$-|a| \le a \le |a| \text{ and } -|b| \le b \le |b|$$

Let's add these two together. We get

$$\begin{aligned}
&-|a| \le a \le |a| \quad + \quad -|b| \le b \le |b| \\
&= \ (-|a|)+(-|b|) \le a+b \le |a|+|b| \qquad\qquad \text{(B)}
\end{aligned}$$

Since a and b are real numbers, adding them will also result in a real number. Well, there are two types of real numbers. What are they?? Remember!!!!! They are either > = 0, or they are < 0! Ok!!

SO we have

$$a+b \ge 0 \text{ or } a+b < 0$$

In the first of these cases where $a+b \ge 0$ certainly $a+b = |\ a+b\ |$

by definition of absolute value. so the right-hand inequality in (B) gives

$$|a+b| \le |a|+|b|$$

In the second case

$$|a+b| = -(a+b)$$

But this is the same as

$$a+b = -|a+b|$$

So the left-hand inequality in (B) can be written as

$$-(\,|\,a\,|+|\,b\,|\,) \le -|\,a+b\,|$$

Multiplying both sides of this inequality by - 1 give

$$|a+b| \le |a| + |b|$$