Lecture # 19

Implicit Differentiation

- The method of Implicit Differentiation
- Derivatives of Rational Powers of x
- Differentiability of Implicit functions

Implicit differentiation

Consider this equation. We want to find its derivative or in other words. But how will we find it if y is not alone on one side of the equation?
Take x y = 1
One way is to solve this equation first to get y
y = 1/x
Differentiating on both sides

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x)^{-1}$$

$$\frac{dy}{dx} = -\frac{1}{x^2}$$

We know that this is the derivative because we used the POWER Rule to differentiate
In this example it was possible to solve the equation for y. What if we cant in some example? Let's see if we can find the derivative in this example without solving for y.

$$x\frac{d}{dx}(y) + y\frac{d}{dx}(x) = 0$$

$$x\frac{dy}{dx} + y(1) = 0$$

$$x\frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

Here i am treating y as an unknown function of x

$$\frac{dy}{dx} = -\frac{y}{x} \qquad \text{Remeber that } xy = 1 \Rightarrow y = \frac{1}{x}$$

$$\text{So, } \frac{dy}{dx} = -\frac{1}{x^2} \text{ same as in the first case}$$

So here was a different way of finding the derivative
In this example, we found dy/dx without solving for y first. This is called **IMPLICIT DIFFERENTIATION**
This is used mostly when it is inconvenient or impossible to separate the y or the dependent variable on one side.

Example

$$\text{Find } \frac{dy}{dx} \text{ if } 5y^2 + \sin y = x^2$$

Hard to separate the y variable on one side in this case in order to find the derivative of this function.
Use Implicit differentiation

$$\frac{d}{dx}(5y^2 + \sin y) = \frac{d}{dx}(x^2)$$

$$5\frac{d}{dx}\left(y^2\right) + \frac{d}{dx}\left(\sin y\right) = 2x$$

$$5\left(2y\frac{dy}{dx}\right) + \cos y \,.\, \frac{dy}{dx} = 2x$$

Chain rule here because y is to be treated as an unknown function of x.

$$\frac{dy}{dx}\left(10y + \cos y\right) = 2x$$

$$\frac{dy}{dx} = \frac{2x}{10y + \cos y}$$

The formula for the derivative involves both x and y and they cannot be separated by using algebraic rules.
Since the original equation cannot be solved for y either, the derivative formula must be left like this

Example

Find the slope of the tangent line at the point (4,0) on the graph of

$$7y^4 + x^3y + x = 4$$

To find the slope, we must find dy/dx.
We will use implicit differentiation because the original equation is hard to solve for y.

$$\frac{d}{dx}\left[7y^4 + x^3y + x\right] = \frac{d}{dx}(4)$$

$$\frac{d}{dx}(7y^4) + \frac{d}{dx}(x^3y) + \frac{d}{dx}(x) = 0$$

$$28y^3\frac{dy}{dx} + \left(x^3\frac{dy}{dx} + y\frac{d}{dx}\left(x^3\right)\right) + 1 = 0$$ Using Product Rule and the Chain Rule

$$28y^3\frac{dy}{dx} + x^3\frac{dy}{dx} + 3yx^2 + 1 = 0$$

$$\frac{dy}{dx} = -\frac{3yx^2 + 1}{28y^3 + x^3}$$

We want to find the slope of the tangent line at the point (4,0) So we have $x = 4$, We want to find the slope of the tangent line at the point (4,0)
So we have x = 4, $y = 0$, so

$$m_{\tan} = \left.\frac{dy}{dx}\right|_{\substack{x=4\\y=0}} = -\frac{1}{64}$$

Example

Find $\dfrac{d^2y}{dx^2}$ if $4x^2 - 2y^2 = 9$

Differentiating both sides implicitly gives

$$8x-4y\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{2x}{y}$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{2x}{y}\right)$$

$$\frac{d^2y}{dx^2}=\frac{y(2)-(2x)\frac{dy}{dx}}{y^2}=\frac{2y-2x\left(\frac{2x}{y}\right)}{y^2}$$

$$=\frac{2y^2-4x^2}{y^3}$$

From the original equation we get finally

$$\frac{d^2y}{dx^2}=-\frac{9}{y^3}$$

Derivatives of Rational Powers of x

We saw earlier that the power rule for differentiation holds for ALL Integers

$$\frac{d}{dx}\left[x^n\right]=nx^{n-1}$$

Now we want to expand it to powers that involve Rational numbers

$$\frac{d}{dx}\left[x^r\right]=rx^{r-1}$$

Where r is a rational number