Lecture # 18

The Chain Rule

- Derivative of Composition of Functions (Chain Rule)
- Generalized Derivative formula
- More Generalized Derivative formula
- An Alternative approach to using Chain Rule

Derivative of Composition of Functions (Chain Rule)

Suppose we have two functions *f* and *g* and we know their derivatives. Can we use this information to find the derivative of the composition

$$(f \circ g)(x) = f(g(x))$$

It turns out that we can by a rule call the CHAIN RULE for differentiation . Look at

$y = (f \circ g)(x) = f(g(x))$

Let us introduce the equation u = g(x). Then the first one becomes

$$y = (f \circ g)(x) = f(g(x)) = f(u)$$

We want to use the known things:

$\frac{dy}{du} = f'(u)$ and $\frac{du}{dx} = g'(u)$

To find the derivative

$\frac{dy}{dx} = \frac{d}{dx} f(g(x))$

Here is the way to do it

Theorem 3.5.2 Chain Rule

If g is differentiable at the point x and f is differentiable at the point g(x), then the composition f (g (x)) is differentiable at the point x. Moreover, if

$y = f(g(x))$ and $u = g(x)$, then $y = f(u)$

and

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

So should we prove this? Well, Let's leave this as an exercise for the students. Its not too difficult, but may be a bit lengthy. It is given in Section III of Appendix C of the textbook.

Example

Find

$\frac{dy}{dx}$ if $y = 4\cos(x^3)$

Let $u = x^3$ so that

$y = 4\cos(u)$

By the Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx} = \frac{d}{du}[4\cos(u)] . \frac{d}{dx}[x^3]$$
$$= (-4\sin(u))(3x^2) = (-4\sin(x^3))(3x^2)$$
$$= -12x^2 \sin(x^3)$$

This formula for finding the derivative of a composition of function is easy to remember if you think of canceling the *du* on the top and the bottom resulting in dy / dx!! This is only a technique to remember, this does not
actually happen.

Generalized Derivative formula
The formula we saw for finding the derivative of composition of functions is a little cumbersome. Here is a simpler one.

The chain rule is $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Now y = f(u) gives upon differentiation w..r.t u

$$\frac{dy}{du} = f'(u)$$

Using this in the equation of the chain rule gives

$$\frac{dy}{dx} = \frac{d}{dx}[f(u)] = f'(u)\frac{du}{dx}$$

Powerful formula: Simple and effective
Example

$$f(x) = (x^2 - x + 1)^{23}$$

Let $u = x^2 - x + 1$, so $f(x)$ becomes

$$f(u) = u^{23}$$

Now we apply the new formual we just got to $f(u)$ to get

$$\frac{d}{dx}[(x^2 - x + 1)^{23}] = \frac{d}{dx}[u^{23}] = 23u^{22} \cdot \frac{du}{dx}$$
$$= 23(x^2 - x + 1)^{23} . \frac{d}{dx}(x^2 - x + 1)$$
$$= 23(x^2 - x + 1)^{23} . (2x - 1)$$

Note that this formula involves derivative of functions which have a different independent variable than the variable we are "differentiating with respect to!"

Note that we had in our last example

$$\frac{d}{dx}[u^{23}] = 23u^{22}\frac{du}{dx}$$

Let $u = x$. Then we get

$$\frac{d}{dx}[x^{23}] = 23x^{22}\frac{d}{dx}(x) = 23x^{22}$$

This matches up with what we have seen before. So this formula is a generalization of our differentiation ideas from previous lectures

Here is a table for your reference

Table 3.5.1

$\frac{d}{dx}[u^n] = nu^{n-1}\frac{du}{dx}$	$\frac{d}{dx}[\sqrt{u}] = \frac{1}{2\sqrt{u}}\frac{du}{dx}$
$\frac{d}{dx}[\sin u] = \cos u\frac{du}{dx}$	$\frac{d}{dx}[\cos u] = -\sin u\frac{du}{dx}$
$\frac{d}{dx}[\tan u] = \sec^2 u\frac{du}{dx}$	$\frac{d}{dx}[\cot u] = -\csc^2 u\frac{du}{dx}$
$\frac{d}{dx}[\sec u] = \sec u \tan u\frac{du}{dx}$	$\frac{d}{dx}[\csc u] = -\csc u \cot u\frac{du}{dx}$

Example

$$\frac{d}{dx}[\sin(2x)]$$

Let $u = 2x$. Then using formula A, we have

$$\frac{d}{dx}[\sin(2x)] = \frac{d}{dx}[\sin(u)] = \cos(u)\frac{du}{dx} = \cos(2x).2 = 2\cos(2x)$$

Example

$$\frac{d}{dx}[\tan(x^2+1)]$$

Let $u = x^2+1$ in formula A, then we get

$$\frac{d}{dx}[\tan(x^2+1)] = \frac{d}{dx}\tan(u) = \sec^2(u)\frac{du}{dx} = \sec^2(x^2+1)\cdot 2x$$

$$= 2x\cdot\sec^2(x^2+1)$$

How do we know what to let u equal?

Well, you make your substitution so that the result comes out to be a function that you already know how to differentiate. Like in the last one, we made it so that we got tan(u) which is easy to differentiate.

Example

$$\frac{d}{dx}[\sqrt{x^3+\cos ec(x)}]$$

Let $u = x^3+\cos ec(x)$. Then we get from A

$$\frac{d}{dx}[\sqrt{x^3+\cos ec(x)}] = \frac{d}{dx}[\sqrt{u}] = \frac{1}{2\sqrt{u}}\frac{du}{dx}$$

$$= \frac{1}{2\sqrt{x^3 + \cos ec(x)}} \cdot \frac{d}{dx}[x^3 + \cos ec(x)]$$

$$= \frac{1}{2\sqrt{x^3 + \cos ec(x)}}(3x^2 - \cos ec(x)\cot(x)]$$

An alternative approach to using Chain Rule

Remember that we started with f(g(x)) and then labeled g(x) as u by u = g(x). If we don't do this then we get

$$\frac{d}{dx}[f(g(x))] = f'(g(x)).g'(x)$$

With this notation, we can say informally that the chain rule says
"Derivative of the OUTER function *f*, then Derivative of the INNER function *g*, and multiply the two together"

Example

$$\frac{d}{dx}[\cos(3x+1)]$$

Here, $f(x) = \cos(x),\ g(x) = 3x+1$. So

$$\frac{d}{dx}[\cos(3x+1)] = [\cos(3x+1)]' \cdot (3x+1)' = -\sin(3x+1).3 = -3\sin(3x+1)$$