**Lecture # 17**

## Derivatives of Trigonometric functions

- Derivative of f(x) =sin(x)
- Derivative of f(x) =cos(x)
- Derivative of f(x) =tan(x)
- Derivative of f(x) =sec(x)
- Derivative of f(x) =cosec(x)
- Derivative of f(x) =cot(x)
- Derivative of the functions made of above functions

**Derivative of f (x) =sin (x)**

- We want to find the derivative of sin (x) or to differentiate sin (x).
- By definition of derivative we have the following calculations

$$\frac{d}{dx}(\sin x) = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

$$= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}$$

$$= \lim_{h\to 0}\frac{\sin(x)\cos(h)-\sin(x)+\cos(x)\sin(h)}{h}$$

$$= \lim_{h\to 0}\left[\sin(x)\left(\frac{\cos(h)-1}{h}\right)+\cos(x)\left(\frac{\sin(h)}{h}\right)\right]$$

$$= \lim_{h\to 0}\left[\cos(x)\left(\frac{\sin(h)}{h}\right)-\sin(x)\left(\frac{1-\cos(h)}{h}\right)\right]$$

In sin (x) and cos (x) don't involve h, they are constant as $h \to 0$

$$\lim_{h\to 0}\sin(x) = \sin(x)$$

$$\lim_{h\to 0}\cos(x) = \cos(x)$$

And so

$$\frac{d}{dx}\sin(x) = \cos(x)\lim_{h\to 0}\left(\frac{\sin h}{h}\right) - \sin(x)\lim_{h\to 0}\left(\frac{1-\cos h}{h}\right)$$

$$= \cos(x)(1) - \sin(x)(0) = \cos(x)$$

So we have proved that

$$\frac{d}{dx}\mathbf{\sin(x)} = \mathbf{\cos(x)}$$

**Derivative of f (x) = cos (x)**

In the same way we can find the derivative of the cos function

$$\frac{d}{dx}\cos(x) = \lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h}$$

This is what we get from the definition of the derivative. The student can work out the details of the calculations here!

**Derivative of f (x) =tan (x)**

We can use the definition of derivative to get

$$\frac{d}{dx}\tan(x) = \lim_{h\to 0}\frac{\tan(x+h)-\tan(x)}{h}$$

I don't recall the expansion for tan (x +h)!! However, we can use the identity

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

And expand it

$$\tan(x+h) = \frac{\sin(x+h)}{\cos(x+h)} = \frac{\sin(x)\cos(h)+\sin(h)\cos(x)}{\cos(x)\cos(h)-\sin(x)\sin(h)}$$

So we get

$$\frac{d}{dx}\tan(x) = \lim_{h\to 0}\frac{\tan(x+h)-\tan(x)}{h}$$

$$= \lim_{h\to 0}\frac{\dfrac{\sin(x+h)}{\cos(x+h)}-\dfrac{\sin(x)}{\cos(x)}}{h}$$

$$= \lim_{h\to 0}\frac{\left[\dfrac{\sin(x)\cos(h)+\sin(h)\cos(x)}{\cos(x)\cos(h)-\sin(x)\sin(h)}\right]-\dfrac{\sin(x)}{\cos(x)}}{h}$$

BIG Formula!!!

I will leave to the student to solve this and get the derivative. But here is what I will do. A simpler way of finding the derivative of tan(x).

Remember the Quotient Rule from precious lectures?? Well, we can use it here instead of the definition of Derivative for tan(x).

Here is how

$$\frac{d}{dx}\tan(x) = \frac{d}{dx}\left[\frac{\sin(x)}{\cos(x)}\right] = \frac{\cos(x)\dfrac{d}{dx}\sin(x)-\sin(x)\dfrac{d}{dx}\cos(x)}{\cos^2(x)}$$

$$= \frac{\cos(x)\cos(x)-\sin(x)[-\sin(x)]}{\cos^2(x)} = \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)}$$

$$= \frac{1}{\cos^2(x)} = \sec^2(x)$$

We used the quotient rule and the derivatives of sin (x) and cos(x)

**Derivative of f(x) =sec(x)**

$$\frac{d}{dx}\sec(x)=\frac{d}{dx}\left(\frac{1}{\cos(x)}\right)=\frac{\cos(x)(0)-(1)[-\sin(x)]}{\cos^2(x)}$$

$$=\frac{\sin(x)}{\cos(x)}\cdot\frac{1}{\cos(x)}=\sec(x)\tan(x)$$

**Derivative of f(x) =cosec(x)**

$$\frac{d}{dx}\operatorname{cosec}(x)=\frac{d}{dx}\left(\frac{1}{\sin(x)}\right)=\frac{\sin(x)(0)-(1)[\cos(x)]}{\sin^2(x)}$$

$$=\frac{-\cos(x)}{\sin(x)}\cdot\frac{1}{\sin(x)}=-\operatorname{cosec}(x)\cot(x)$$

**Derivative of f(x) =cot(x)**

$$\frac{d}{dx}\cot(x)=\frac{d}{dx}\left(\frac{1}{\tan(x)}\right)=\frac{\tan(x)(0)-(1)[\sec^2(x)]}{\tan^2(x)}$$

$$=\frac{-\sec^2(x)}{\tan^2(x)}=-\cos ec^2(x)$$

**Example**

Suppose that the rising sun passes directly over a building that is 100 feet high and let $\theta$ be the angle of elevation of the sun. Find the rate at which the length $x$ of the building's shadow is changing with respect to $\theta$

When $\theta=45^{\circ}$. Express the answer in units of feet/degree.

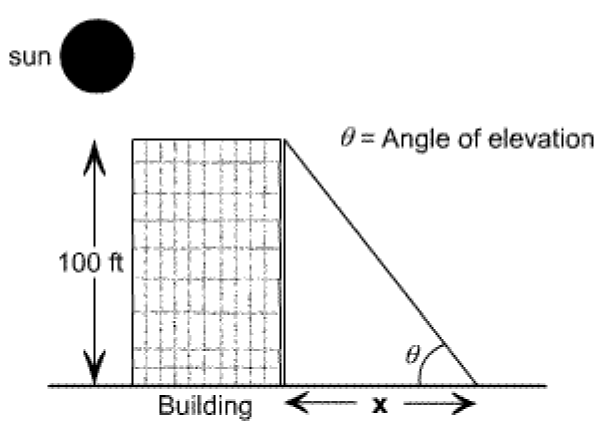
**Solution**

From the figure, we see that the variable $\theta$ and $x$ are related by the equation

$$\tan\theta=\frac{100}{x}\Rightarrow x=100\cot$$

We want to find the Rate of Change of $x$ wrt $\theta$ or in other words

$$\frac{dx}{d\theta}=?$$




$$\tan\theta=\frac{100}{x}$$

$$x=100\cot\theta$$

I would like to use the fact we got earlier that

$$\frac{d}{dx}\cot(\theta) = -\cos ec^2(\theta)$$

This will work only if theta is defined in RADIANS. WHY, because we want cot to be a function which is defined in terms of radians.

We can do that here and instead of degrees, use radians to measure theta. So 45 deg will become radians. $\frac{\pi}{4}$

So we get

$$\frac{dx}{d\theta} = -100\cos ec^2\theta$$

This is the rate of change of the length $x$ of shadow wrt to the elevation angle theta in units of feet/radian. When theta is $\frac{\pi}{4}$ radians, then

$$\left.\frac{dx}{d\theta}\right|_{\theta=\frac{\pi}{4}} = -100\cos ec^2(\frac{\pi}{4}) = -200\,feet/radian$$

Now we want to go back to degrees because we were asked to answer the question with the angle in degrees. We have the relationship

$$180 \text{ degrees} = \pi \text{ radians}$$

$$1 \text{ degree} = \frac{\pi}{180}\text{radian} \Rightarrow \text{ There are } \frac{\pi}{180}\text{radian/degree}$$

This Gives

$$\text{-200 feet/radian } \cdot \frac{\pi}{180}\text{radians/degree} = -\frac{10}{9}\pi \text{ feet/degree}.$$