Lecture # 16

Techniques Of Differentiation

- In the lectures so far, we obtained some derivatives directly by definition.
- In this sections, we develop theorems which will give us short cuts for calculation derivatives of special functions
- Derivatives of Constant Functions
- Derivatives of Power functions
- Derivative of a constant multiple of a function Etc!!

Derivatives of Constant Functions
Theorem 3.3.1
If "f" is a constant function f(x)=c for all x, then

$$f'(x)=\frac{d}{dx}[f(x)]=\frac{d}{dx}[c]=0$$

Proof

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Since f(x) = c so $\lim_{h\to 0} f(x+h) = c$

$$=\lim_{h\to 0}\frac{c-c}{h} = \lim_{h\to 0}\frac{0}{h}$$

$$=\lim_{h\to 0} 0$$

$$=0$$

This result is also obvious geometrically since the function y = c is a horizontal line with slope 0. And we saw earlier that a line function has tangent line slope equal to its own slope which is 0.

Example
f(x)=5 so $f'(x)=0$

Theorem 3.3.2 (Power Rule)
If n is a positive integer, then

$$\frac{d}{dx}[x^n]=n.x^{n-1}$$

Proof
Let $f(x)=x^n$, n is a positive integer. Then

$$\frac{d}{dx}[x^n]= f`(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{(x+h)^n-x^n}{h}$$

Using the binomial theorem on $(x + h)^n$

$$f'(x) =$$

$$\lim_{h\to 0}\frac{\left[x^n + nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + ... + nxh^{n-1} + h^n\right] - x^n}{h}$$

$$= \lim_{h\to 0}\frac{nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + ... + nxh^{n-1} + h^n}{h}$$

Take h common from numerator and cancel with the denominator to get

$$= \lim_{h\to 0}[nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}h + ... + nxh^{n-2} + h^{n-1}]$$

Distributing the limit over the sum give all the terms equal to zero except the first one.
So

$$f`(x) = nx^{n-1}$$

Example

$$\frac{d}{dx}[x^5] = 5x^4, \quad \frac{d}{dx}[x] = 1x^{1-1} = 1x^0 = 1\cdot 1 = 1$$

Theorem 3.3.3
Let c be a constant and f be a function differentiable at x, then so is the function c.f and

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$$

Proof

$$\frac{d}{dx}[cf(x)] = \lim_{h\to 0}\frac{cf(x+h) - cf(x)}{h}$$

$$= \lim_{h\to 0} c\left[\frac{f(x+h) - f(x)}{h}\right]$$

$$= c\lim_{h\to 0}\left[\frac{f(x+h) - f(x)}{h}\right]$$

$$= c\frac{d}{dx}[f(x)]$$

Example

$$\frac{d}{dx}[3x^8]$$

$$= 3\frac{d}{dx}[x^8]$$

$$= 3(8x^7) \quad 1x^0 = 1\cdot 1 = 1$$

$$= 24x^7 \quad nx^{n-1}$$

Derivative of Sums and Differences of Functions

If f and g are differentiable functions at x, then so is f+g, and

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

Proof

$$\frac{d}{dx}[[f(x)+g(x)] = \lim_{h\to 0}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h}$$

$$= \lim_{h\to 0}\frac{[f(x+h)-f(x)]+[g(x+h)-g(x)]}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$== \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

Similarly for Difference. Left as an exercise.

Example

$$\frac{d}{dx}[x^4 + x^3]$$

$$= \frac{d}{dx}[x^4] + \frac{d}{dx}[x^3]$$

$$= 4x^3 + 3x^2$$

In general

$$\frac{d}{dx}[f_1(x) + f_2(x) + \ldots + f_n(x)]$$

$$= \frac{d}{dx}[f_1(x)] + \frac{d}{dx}[f_2(x)] + \ldots + \frac{d}{dx}[f_n(x)]$$

Derivative of a Product

Theorem 3.3.5

If f and g are differentiable functions at x, then so is f.g and

$$\frac{d}{dx}[f(x).g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

Proof

$$\frac{d}{dx}[f(x)g(x)] = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

If we add and subtract f(x + h)g(x) in the numerator

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\left[f(x+h)\frac{g(x+h)-g(x)}{h} + g(x)\frac{f(x+h)-f(x)}{h}\right]$$

$$= \lim_{h\to 0} f(x+h).\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$+ \lim_{h\to 0} g(x).\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} f(x+h).\frac{d}{dx}[g(x)] + \lim_{h\to 0} g(x).\frac{d}{dx}[f(x)]$$

Since $\lim_{h\to 0} g(x) = g(x)$ becasue there is no h involved

and $\lim_{h\to 0} f(x+h) = f(x)$, we have the desired result

$$= f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$$

$$(f.g)' = f.g' + g.f'$$

Example

$$\frac{d}{dx}[(4x^2)(3x)]$$

$$= 4x^2\frac{d}{dx}(3x) + 3x\frac{d}{dx}(4x^2)$$

$$= 4x^2(3) + 3x(8x) = 12x^2 + 24x^2 = 36x^2$$

Derivative of Quotient

Theorem 3.3.6

If f and g are differentiable functions at x, and $g(x) \neq 0$. Then f/g is differentiable at x and

$$\frac{d}{dx}[\frac{f(x)}{g(x)}] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

Prove Yourself!

$$\left(\frac{f}{g}\right)' = \frac{g.f' - f.g'}{g^2}$$

Example

$$\frac{d}{dx}\left(\frac{3x}{5x^2}\right)$$

$$=\frac{5x^2\frac{d}{dx}(3x)-3x\frac{d}{dx}(5x^2)}{(5x^2)^2}$$

$$=\frac{5x^2(3)-3x(10x)}{25x^4}$$

$$=\frac{15x^2-30x^2}{25x^4}=\frac{-15x^2}{25x^4}=\frac{-3}{5x^2}$$

Derivative of a Reciprocal
Theorem 3.3.7

If g is differentiable at x, and $g(x)\neq 0$,the $\frac{1}{g(x)}$ is differentiable at x and

$$\frac{d}{dx}[\frac{1}{g(x)}]=-\frac{\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

Student can prove this using the quotient rule!!
Using the Reciprocal Theorem we can generalize Power Rule (Theorem3.3.1) for all integers (negative or non-negative)

Theorem 3.3.8

If n is any integer, then $\frac{d}{dx}[x^n]=x^{n-1}$