Lecture 15

The Derivative

In the previous lecture we saw that the slope of a tangent line to the graph of y = f (x) is given by

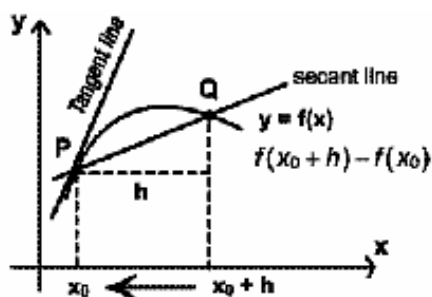$$m_{tan} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Let's do some algebraic manipulations.   Let
$h = x_1 - x_0$ so that $x_1 = x_0 + h$ and $h \to 0$ as $x_1 \to x_0$.

So we can rewrite the above tangent formula as

$$m_{tan} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Definition 3.2.1

If P(x_0, y_0) is a point on the graph of a function f then the tangent line to the graph of f at P is defined to be the line through P with slope

$$m_{tan} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Tangent line at $P(x_0, y_0)$ is just called the tangent line at x_0 for brevity. Also a point $P(x_0, y_0)$ make here that the Equation. We make this definition provided that the LIMIT in the definition exists! Equation of the tangent line at the point $P(x_0, y_0)$ is

$$y - y_0 = m_{tan}(x - x_0)$$

Example

Find the slope and an equation of the tangent line to the graph of $f(x) = x^2$ at the point P(3,9). Here is the

$f(x) = x^2$

P(3,9)
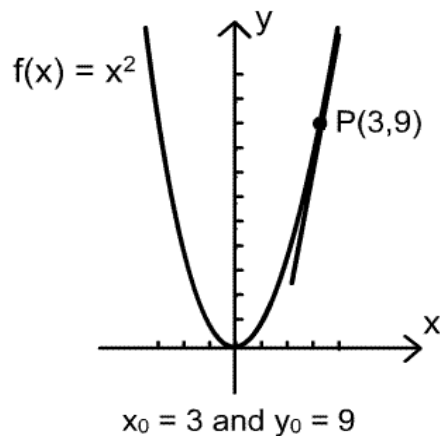
$x_0 = 3$ and $y_0 = 9$

We use the formula given in the above definition with $x_0 = 3$ and $y_0 = 9$.

First we find the slope of the tangent line at $x_0 = 3$

$$m_{tan} = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0} \frac{(3+h)^2 - 9}{h}$$

$$= \lim_{h \to 0} \frac{(9 + 6h + h^2) - 9}{h} = \lim_{h \to 0} \frac{6h + h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(6+h)}{h} = \lim_{h \to 0} (6+h) = 6$$

Now we find the equation of the tangent line

$$y - 9 = 6(x - 3)$$
$$\Rightarrow y = 6x - 9$$

Now notice that m_{tan} is a function of x_0 because since it depends on where along the curve is being computed. Also, from the formula for it, it should be clear that h eventually shrinks to 0 and whatever is left will be in terms of x_0. This can be further modified by saying that we will call x_0 is x. Then we have m_{tan} as a function of x and this is nice.

Since now we can say that we have associated a new function m_{tan} to any given function. We can rewrite the formula for m_{tan} as

$$m_{\tan} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This is a function of x and its very important. Its called the Derivative function with respect to x for the function $y = f(x)$

Definition 3.2.2

The function f defined by the formula

$$f' = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

is called the derivative with respect to x of the function f. The domain of f' consists of all x for which limit exists.

We can interpret this derivative in 2 ways **Geometric interpretation of the Derivative** f' is the function whose value at x is the slope of the tangent line to the graph of the function f at x **Rate of Change is an interpretation of Derivative.** If y = f(x), then f' is the function whose value at x is the instantaneous rate of change of y with respect to x at the point x.

Example

Let $f(x) = x^2 + 1$

Find $f'(x)$.

Use the result from part a) to find the slope of the tangent line to

$y = f(x) = x^2 + 1$

$$f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[(x+h)^2 + 1] - [x^2 + 1]}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 1 - x^2 - 1}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h \to 0} (2x + h) = 2x$$

we show that the slope of the tangent line at ANY point x is $f'(x) = 2x$, So at point $x = 2$ we have slope $f'(2) = 2(2) = 4$ at point $x = 0$ we have slope $f'(0) = 2(0) = 0$ at point $x = -2$ we have slope $f'(-2) = 2(-2) = -4$

Example 3

It should be clear that at each point on a straight line $y = mx + b$ the tangent line coincides with the line itself. So the slope of the tangent line must be the same as that of the original line, namely m. We can prove this here
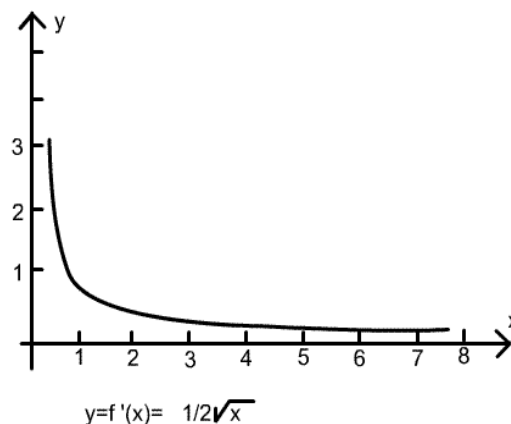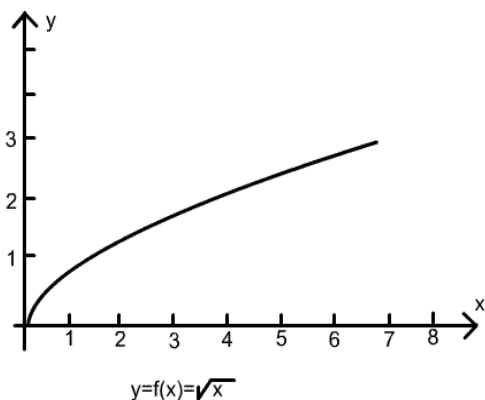
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[m(x+h) + b] - (mx + b)}{h}$$

$$= \lim_{h \to 0} \frac{mx + mh + b - mx - b}{h} = \lim_{h \to 0} \frac{mh}{h} = \lim_{h \to 0} m = m$$

Find the derivative with respect to x of $f(x) = \sqrt{x}$

$$f(x) = \sqrt{x}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Here are the graphs of f (x) and its derivative we just found. Note that

$$\lim_{x \to 0^+} \frac{1}{2\sqrt{x}} = +\infty$$

y=f(x)=√x

y=f '(x)= 1/2√x

the derivative of graph shows that as x goes to 0 from the right side, the slopes of the tangent lines to the graph of y = f (x) approach +inf, meaning that the tangent lines start getting VERTICAL!! Can you see this??!

Derivative Notation

The process of finding the derivative is called DIFFERENTIATION.

It is useful often to think of differentiation as an OPERATION that is applied to a given function to get a new one f'. Much like an arithmetical operation +

In case where the independent variable is x the differentiation operation is written as

This is read as "the derivative of f with respect to x $\dfrac{d}{dx}[f(x)]$

So we are just giving a new notation for the same idea but this will help us when we want to think of Derivative or Differentiation from a different point of view $\dfrac{d}{dx}[f(x)] = f'(x)$.

With this notation we can say about the previous example

$$\frac{d}{dx}[\sqrt{x}] = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}[\sqrt{x}]\bigg|_{x=x_0} = \frac{1}{2\sqrt{x}}\bigg|_{x=x_0} = \frac{1}{2\sqrt{x_0}}$$

If we write $y = f(x)$, then we can say

$$\frac{d}{dx}[y] = f'(x)$$

$$\frac{dy}{dx} = f'(x)$$

So we could say for the last example

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

This looks like a RATIO, and later we will see how this is true in a certain sense. But for now $\dfrac{dy}{dx}$ should be regarded as a single SYMBOL for the derivative of a function $y = f(x)$.

If the independent variable is not x but some other variable, then we can make appropriate adjustments. If it is $u,$ then

$$\frac{dy}{du} = f'(u) \quad \text{and} \quad \frac{d}{du}[f(u)] = f'(u)$$

One more notation can be used when one wants to know the value of the derivative a a certain point $x = x_0$

$$\frac{d}{dx}[f(x)]\Big|_{x=x_0} = f'(x_0)$$

For example

$$\frac{d}{dx}[\sqrt{x}]\Big|_{x=x_0} = \frac{1}{2\sqrt{x}}\Big|_{x=x_0} = \frac{1}{2\sqrt{x_0}}$$

Existence of Derivatives

From the definition of the derivative, it is clear that the derivative exists only at the points where the limit exists.

If x_0 is such a point, then we say that f **is differentiable at** x_0 OR f **had a derivative at** x_0. This basically defines the domain of f' as those points x at which f is differentiable

f is **differentiable on an open interval (a,b)** if it is differentiable at EACH point in (a,b). f is **differentiable function** if its differentiable on the interval. The points at which f is not differentiable, we will say **the derivative of f does not exist at those points.**

Non differentiability usually occurs when the graph of f(x) has

- **corners**

- **Vertical tangents**
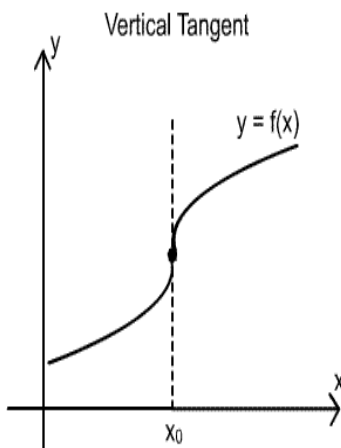
- **Points of discontinuity**

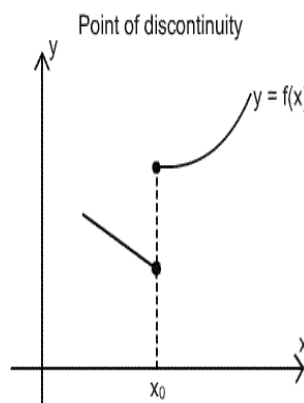Let's look at each case and get a feel for why this happens
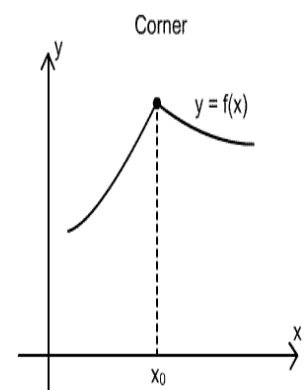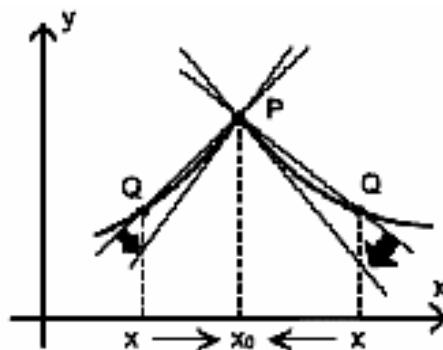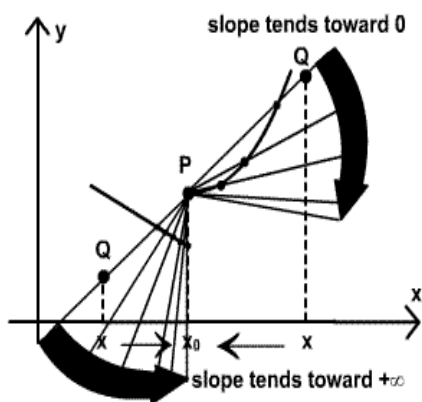

Fig 3.2.6(b)   Fig 3.2.6(c)   Fig 3.2.6(a)

At corners, the two sided limits don't match up when we take the limit of the secant lines to get the slope of the tangents

- At points of discontinuity also we have the two sided limits not agreeing and therefore the function is not differentiable.

- Vertical tangents occur when the slope of the tangent line approaches to $\pm\infty$ as we take the limit of the secant line's slope
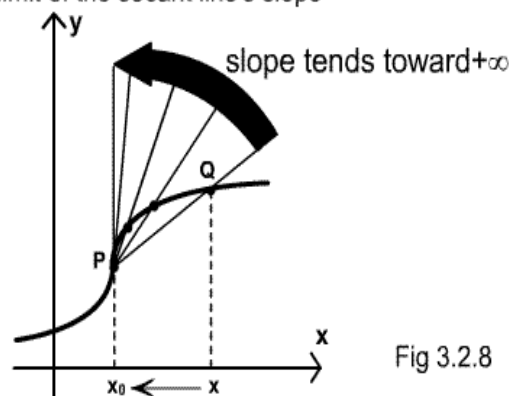

Fig 3.2.8

Relationship between Differentiability and Continuity

Theorem 3.2.3

If f is differentiable at a point x_0 then f is also continuous at x_0.

Proof

We will use this definition we saw earlier of continuity $\lim\limits_{h \to 0} f(h + x_0) = f(x_0)$ Where

x_0 is any point. So we will show that

$\lim\limits_{h \to 0} f(x_0 + h) = f(x_0)$ or equivalently,

$\lim\limits_{h \to 0} [f(x_0 + h) - f(x_0)] = 0$

$$\lim_{h \to 0}[f(x_0 + h) - f(x_0)] = \lim_{h \to 0}\left[\frac{f(x_0 + h) - f(x_0)}{h} \cdot h\right]$$

$$= \lim_{h \to 0}\left[\frac{f(x_0 + h) - f(x_0)}{h}\right] \cdot \lim_{h \to 0} h$$

$$= f'(x_0).0 = 0$$

So this theorem says that a function cannot be differentiable at a point   of discontinuity

Example

$$f(x) = |x|$$

Find $f'(x)$.

Remember that $|x| = \sqrt{x^2}$ .Can you use this to differentiate? Yes, but for this we need more theory and we will see how to do this later

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$f'(x) = \frac{d}{dx}|x| = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$