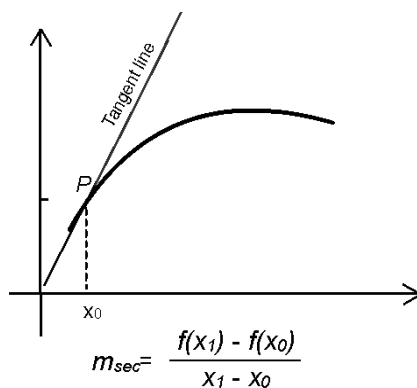


LECTURE 14

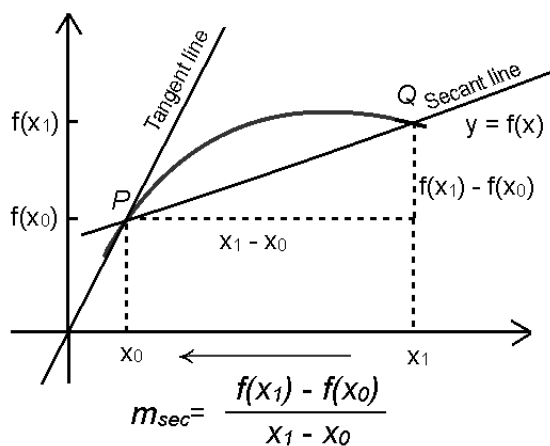
Tangent Lines and Rates of Change

In lecture 9, we saw that a Secant line between two points was turned into a tangent line. This was done by moving one of the points towards the other one. The secant line rotated into a LIMITING position which we regarded as a TANGENT line.



For now just consider Secant lines joining two points on a curve (graph) if a function of the form $y = f(x)$. If $P(x_0, y_0)$ and $Q(x_1, y_1)$ are distinct points on a curve $y = f(x)$, then secant line connecting them has slope

$$m_{sec} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$



If we let $x_1 \rightarrow x_0$ then Q will approach P along the graph of the function $y = f(x)$, and the secant line will approach the tangent line at P.

This will mean that the slope of the Secant line will approach that of the Tangent line at P as $x_1 \rightarrow x_0$, So we have the following

$$m_{\text{tan}} = \lim_{x_1 \rightarrow x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

We just saw how to find the slope of a tangent line.

This was a geometric problem. In the 17th century, mathematicians wanted to define the idea of Instantaneous velocity. This was a theoretical idea. But they realized that this could be defined using the geometric idea of tangents.

Let's define Average velocity formally

$$\text{Average Velocity} = \frac{\text{distance travelled}}{\text{Time Elapsed}}$$

This formula tells us that the average velocity is the velocity at which one travels on average during some interval of time!!

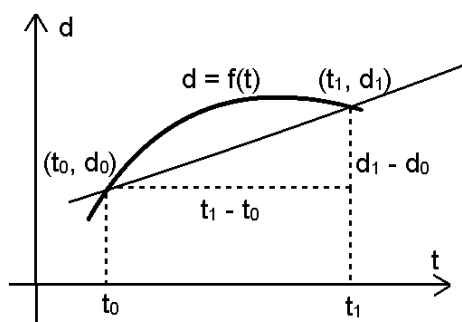
More interesting than Average Velocity is the idea of Instantaneous velocity. This is the velocity that an object is traveling at a given INSTANT in time. When a car hits a tree, the damage is determined by the INSTANTANEOUS velocity at the moment of impact, not on the average speed during some time interval before the impact.

To define the concept of instant velocity, we will first look at distance as a function of time,

$d = f(t)$. After all, distance covered is a physical phenomenon which is always measured with respect to time. Going from New York to San Francisco (km) takes about 6 hours, if your average speed is 800 km/h. This will give us a way to plot the position versus time curve for motion.

Now we will give a geometric meaning to the concept of Average Velocity.

Average velocity is defined as the distance traveled over a given time of period. So if your curve for $f(t)$ looks like as given below



then the average velocity over the time interval $[t_0, t_1]$ is defined as

$$\text{Average Velocity} = \frac{\text{distance traveled during the interval}}{\text{Time Elapsed}}$$

$$\frac{d_1 - d_0}{t_1 - t_0} = v_{ave} = \frac{f(t_1) - f(t_0)}{t_1 - t_0}$$

$d_1 - d_0$ is the distance traveled in the interval.

So average velocity is just the slope of the Secant line joining the points (t_0, d_0) and (t_1, d_1) .

Say we want to know the instantaneous velocity at the point t_0 . We can find this by letting t_1 approach t_0 . When this happens, the interval over which the average velocity is measured shrinks and we can approximate instant velocity.

As t_1 gets very close to t_0 , our approximate instantaneous velocity will get better and better. As this continues, we can see that the average velocity over the interval gets closer to instantaneous velocity

at t_0 . So we can say

$$v_{inst} = \lim_{t_1 \rightarrow t_0} v_{ave} = \lim_{t_1 \rightarrow t_0} \frac{f(t_1) - f(t_0)}{t_1 - t_0}$$

But this is just the slope of the tangent line at the point (t_0, d_0) . Remember that the limit here means that the two sided limits exist.

Average and Instantaneous rates of change

Let's make the idea of average and instantaneous velocity more general. Velocity is the rate of change of position with respect to time. Algebraically we could say:

Rate of change of \mathbf{d} with respect to \mathbf{t} . Where $d = f(t)$.

Rate of change of bacteria w.r.t time.

Rate of change a length of a metal rod w.r.t to temperature

Rate of change of production cost w.r.t quantity produced.

All of these have the idea of the rate of change of one quantity w.r.t another quantity.

We will look at quantities related by a functional relationship $y = f(x)$

So we consider the rate of change of y w.r.t x or in other words, the rate of change of the dependant variable (quantity) w.r.t the Independent variable (quantity).

Average rate of change will be represented by the slope of a certain Secant Line.

Instantaneous rate of change will be represented by the slope of a certain tangent Line.

Definition 3.1.1

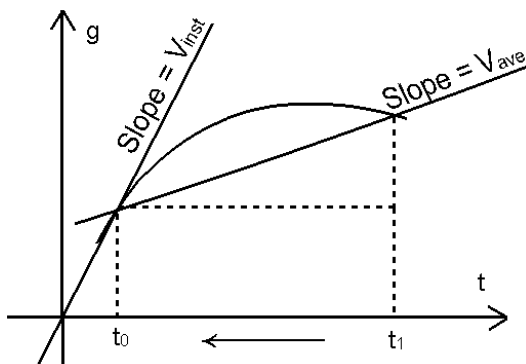
If $y = f(x)$, then the average rate of change of y with respect to x over the interval $[x_0, x_1]$ is the slope m_{sec} of the secant line joining the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ on the graph of f

$$m_{\text{sec}} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

If $y = f(x)$, then the **Average rate of Change of y with respect to x over the interval $[x_0, x_1]$** is the slope of the secant line joining the points $[x_0, f(x_0)]$ and $[x_1, f(x_1)]$. That is

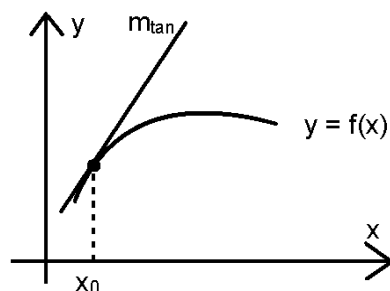
$$m_{\text{sec}} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

And on the graph of f .

**Definition 3.1.1**

If $y = f(x)$, then the **instantaneous rate of Change of y with respect to x at the point x_0** is the slope m_{tan} of the tangent line to graph of f at the point x_0 , that is

$$m_{\text{tan}} = \lim_{x_1 \rightarrow x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$



Example

Let $y = f(x) = x^2 + 1$

- Find the average rate of y w.r.t to x over the interval $[3,5]$
- Find the instantaneous rate of change of y w.r.t x at the $x = x_0$ point $x_0 = -4$
- Find the instantaneous rate of change of y w.r.t x at a general point

Solution:

We use the formula in definition of Average rate with

$$y = f(x) = x^2 + 1, \quad x_0 = 3 \quad \text{and} \quad x_1 = 5$$

$$m_{\text{sec}} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(5) - f(3)}{5 - 3} = \frac{26 - 10}{5 - 3} = 8$$

So y increase 8 units for each unit increases in x over the interval $[3,5]$

b) Applying the formula with $y = f(x) = x^2 + 1$ and $x_0 = -4$ gives

$$\begin{aligned} m_{\text{tan}} &= \lim_{x_1 \rightarrow x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{x_1 \rightarrow -4} \frac{(x_1^2 + 1) - 17}{x_1 + 4} \\ &= \lim_{x_1 \rightarrow -4} \frac{x_1^2 - 16}{x_1 + 4} = \lim_{x_1 \rightarrow -4} (x_1 - 4) = -8 \end{aligned}$$

Negative inst rate of change means its DECREASING

c) Here we have

$$\begin{aligned} m_{\text{tan}} &= \lim_{x_1 \rightarrow x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{x_1 \rightarrow x_0} \frac{(x_1^2 + 1) - (x_0^2 + 1)}{x_1 - x_0} \\ &= \lim_{x_1 \rightarrow x_0} \frac{x_1^2 - x_0^2}{x_1 - x_0} = \lim_{x_1 \rightarrow x_0} (x_1 + x_0) = 2x_0 \end{aligned}$$

The result of part b) can be obtained from this general result by letting.