LECTURE 13

Limits and continuity of Trigonometric functions

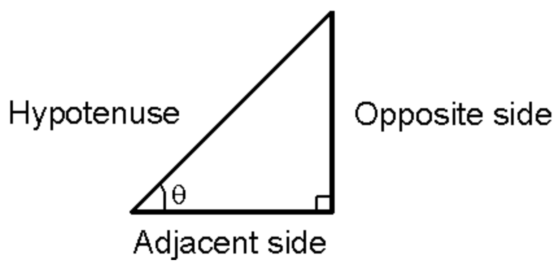
- Continuity of Sine and Cosine functions
- Continuity of other trigonometric functions
- Squeeze Theorem
- Limits of Sine and Cosine as x goes to +- infinity

You will have to recall some trigonometry. Refer to Appendix B of your textbook.

Continuity of Sine and Cosine

Sin and Cos are ratios defined in terms of the acute angle of a right angle triangle and the sides of the triangle. Namely,

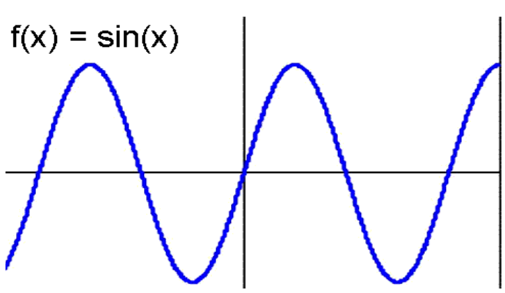
$$\cos\theta = \frac{adjacent\ side}{Hypoteneous} \qquad \sin\theta = \frac{Opposite\ side}{Hypoteneous}$$

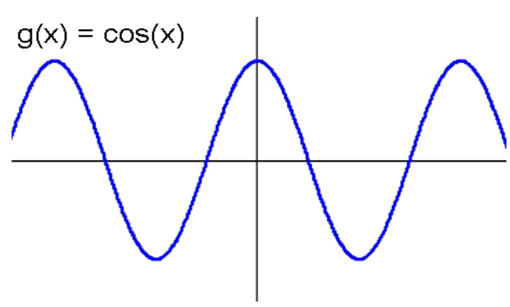


We look at these ratios now as functions. We consider our angles in radians

- Instead of θ we will use x

Here is a picture that shows the graph of f(x) = sin (x). Put the circle picture here, and then unravel it and get the standard picture.



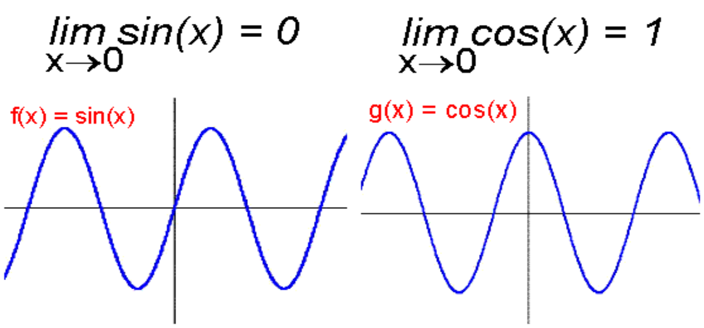


From the graph of Sin and cosine, its obvious that

$$\lim_{x\to 0} \sin(x) = 0 \qquad\qquad \lim_{x\to 0} \cos(x) = 1$$

This is the intuitive approach. Prove this using the Delta Epsilon definitions!!
Note that sin(0) = 0 and cos(0) = 1
Well, the values of the functions match with those of the limits as x goes to 0!! So we have this theorem
THEOREM 2.8.1

The functions sin(x) and cos(x) are continuous. As clear from figure



Here is the definition of continuity we saw earlier.
A function *f* is said to be continuous at *c* if the following are satisfied

(a) f(c) is defined
(b) $\lim_{x\to c} f(x)$ exists
(c) $\lim_{x\to c} f(x) = f(c)$

Let $h = x - c$. So $x = h + c$. Then $x \to c$ is equivalent to the requirement that $h \to 0$. So we have

Definition

A function is continuous at *c* if the following are met

(a) f(c) is defined
(b) $\lim_{h\to 0} f(h+c)$ exists
(c) $\lim_{h\to 0} f(h+c) = f(c)$

We will use this new definition of Continuity to prove

Theorem 2.8.1
The functions sin(x) and cos(x) are continuous.

Proof

We will assume that $\lim_{x \to 0} \sin(x) = 0$ and $\lim_{x \to 0} \cos(x) = 1$

From the above, we see that the first two conditions of our continuity definition are met. So just have to show by part 3) that

$$\lim_{h \to 0} \sin(c+h) = \sin(c)$$

$$\lim_{h \to 0} \sin(c+h) = \lim_{h \to 0} [\sin(c)\cos(h) + \cos(c)\sin(h)]$$
$$= \lim_{h \to 0} \sin(c)\cos(h) + \lim_{h \to 0} \cos(c)\sin(h)$$
$$= \sin(c)\lim_{h \to 0} \cos(h) + \cos(c)\lim_{h \to 0} \sin(h)$$
$$= \sin(c)(1) + \cos(c)(0) = \sin(c)$$

The continuity of cos(x) is also proved in a similar way, and I invite you to try do that!

Continuity of other trigonometric functions

Remember by theorem 2.7.3 that if f(x) and g(x) are continuous , then so is $h(x) = \dfrac{f(x)}{g(x)}$. Except where g(x) = 0. So tan(x) is continuous everywhere except at cos(x) = 0 which gives

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$x = \pm\frac{\pi}{2},\ \pm\frac{3\pi}{2},\ \pm\frac{5\pi}{2},\ldots$$

Likewise, since

$$\cot(x) = \frac{\cos(x)}{\sin(x)},\quad \sec(x) = \frac{1}{\cos(x)}\quad \text{cosec}(x) = \frac{1}{\sin(x)}$$

We can see that they are all continuous on appropriate intervals using the continuity of sin(x) and cos(x) and theorem 2.7.3

Squeeze Theorem for finding Limits

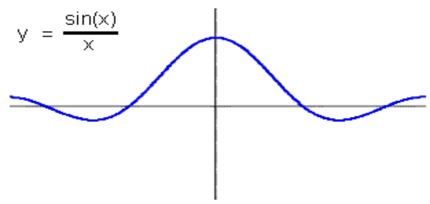
We will show that .These is important results which will be used later. If you remember, the very first example of limits we saw was

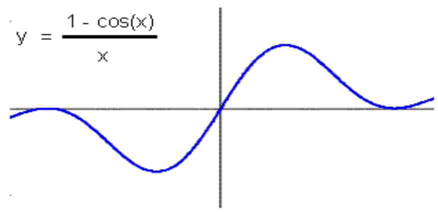
$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1 \quad \text{and} \quad \lim_{x \to 0} \frac{1-\cos(x)}{x} = 0$$

Now we prove this

$$\lim_{x \to 0} \frac{\sin(x)}{x}$$

Here are the graphs of the functions.





They suggest that the limits are what we want them to be! We need to prove this PROBLEM. As x goes to 0, both the top and the bottom functions go to 0. Sin(x) goes to 0 means that the fraction as a whole goes to 0.
x goes to zero means that the fraction as a whole goes to +inf! . There is a tug of war between the Dark Side and the Good Side of the Force.
So there is a tug-of-war between top and bottom.
To find the limit we confine our function between two simpler functions, and then use their limits to get the one we want.

SQUEZZING THEOREM

Let fm g and h be functions satisfying $g(x) \le f(x) \le h(x)$ for all x in some open interval containing the point a, with the possible exception that the inequality need not to hold at a.

We g and h have the same limits as x approaches to a, say
$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$
Then f also has this limit as x approaches to a, that is
$\lim_{x\to a} f(x) = L$.

Example

$\lim_{x\to 0} x^2 \sin^2(\frac{1}{x})$

Remember that the $0 \le \sin(x) \le 1$.

So certainly $0 \le \sin^2(x) \le 1$.

And so $0 \le \sin^2(\frac{1}{x}) \le 1$.

Multiply throughout this last inequality by x^2.
We get

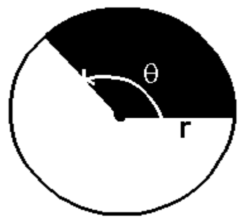
$$0 \le x^2 \sin^2(\frac{1}{x}) \le x^2,$$

But $\lim_{x \to 0} 0 = \lim_{x \to 0} x^2 = 0$

So by the Squeezing theorem

$$\lim_{x \to 0} x^2 \sin^2(\frac{1}{x}) = 0$$

Now Let's use this theorem to prove our original claims. The proof will use basic facts about circles and areas of SECTORS with center angle of θ radians and radius *r*:



The area of a sector is given by $A = \frac{1}{2} r^2 \theta$.

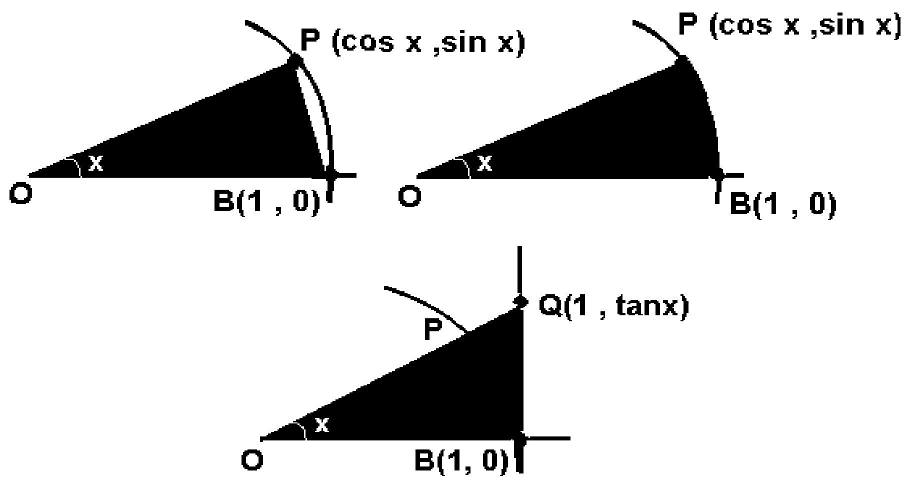
Theorem 2.8.3

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

Proof

Let *x* be such that $0 < x < \frac{\pi}{2}$. Construct the angle *x* in the standard position starting from the center of a unit circle.

We have the following scenario



From the figure we have

$0 <$ area of $\Delta\ OBP <$ area of sector $OBP <$ area of ΔOBQ

Now

area of $\Delta\ OBP = \frac{1}{2} base.height = \frac{1}{2}(1).\sin(x) = \frac{1}{2}\sin(x)$

area of sector $OBP = \frac{1}{2}(1)^2.x = \frac{1}{2}x$

area of $\Delta\ OBQ = \frac{1}{2} base.height = \frac{1}{2}(1)\tan(x) = \frac{1}{2}\tan(x)$

So

$0 < \frac{1}{2}\sin(x) < \frac{1}{2}x < \frac{1}{2}\tan(x)$

Multiplying through by $\frac{2}{\sin(x)}$ gives

$1 < \frac{x}{\sin(x)} < \frac{1}{\cos(x)}$

Taking reciprocals gives

$\cos(x) < \frac{\sin(x)}{x} < 1$

We had made the assumption that $0 < x < \frac{\pi}{2}$.

Also works when $-\frac{\pi}{2} < x < 0$ You can check when you do exercise 4.9 So our last equation holds for all angles x except for $x = 0$.

Remember that $\lim_{x \to 0} \cos(x) = 1$ and $\lim_{x \to 0} 1 = 1$

Taking limit now and using squeezing theorem gives

$$\lim_{x\to 0}\cos(x) < \lim_{x\to 0}\frac{\sin(x)}{x} < \lim_{x\to 0} 1$$

$$= 1 < \lim_{x\to 0}\frac{\sin(x)}{x} < 1$$

Since the middle thing is between 1 and 1, it must be 1!!

Prove yourself that

$$\lim_{x\to 0}\frac{1-\cos(x)}{x} = 0$$

Limits of sin(x) and cos(x) as x goes to +inf or –inf

By looking at the graphs of these two functions its obvious that the y-values oscillate btw 1 and –1 as x goes to +inf or –inf and so the limits DNE!!