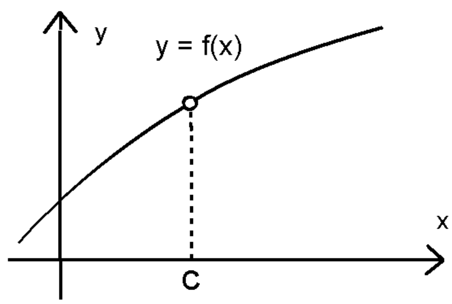
LECTURE 12

Continuity

- Develop the concept of CONTINUITY by examples
- Give a mathematical definition of continuity of functions
- Properties of continuous functions
- Continuity of polynomials and rational functions
- Continuity of compositions of functions
- The Intermediate values theorem

CONTINUITY of a function becomes obvious from its graph at certain points in the plane .We will say CONTINUITY of a function or graph of a function interchangeably.

DISCONTINUITY



The above given curve is discontinuous at point c since f(x) is not defined there.
when the following things happens then there is a break or discontinuity in the graph of a function f(x) at x = c

- f is undefined at c
- The $\lim_{x \to c} f(x)$ does not exist.
- The function is defined at c and the $\lim_{x \to c} f(x)$ exists, but the values of f(x) and the values of the limit differ at the point c
- So we get the following definition for continuity

Definition2.7.1

(a) f(c) is defined

(b) $\lim_{x\to c} f(x)$ exists

(c) $\lim_{x\to c} f(x) = f(c)$

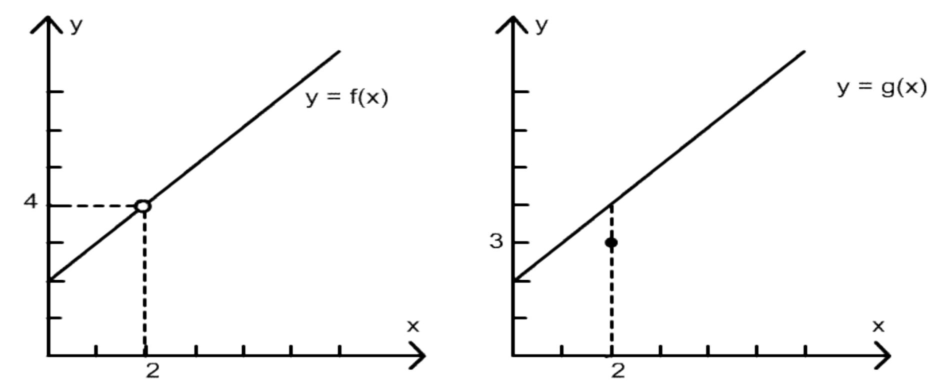
If any of these conditions in this definition fail to hold for a function f(x) at a point c, then f is called discontinuous at c

- c is called the point of discontinuity
- If f(x) is continuous at all points in an interval (a, b), then we say that f is **continuous on (a,b)**
- A function continuous on the interval $(-\infty, +\infty)$ is called a **continuous function**

Example

$$f(x) = \frac{x^2-4}{x-2}$$

$$g(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}$$



f is discontinuous at x = 2 because f(2) is undefined.

g is discontinuous because g(2)=3 and

$$\lim_{x\to 2} g(x) = \lim_{x\to 2}\frac{x^2-4}{x-2} = \lim_{x\to 2}(x+2) = 4$$

So

$$\lim_{x\to 2} g(x) \neq g(2)$$

The last equation does not satisfy the condition of continuity. Condition (3) of the definition is enough to determine whether a function is continuous or not. This is so because if (3) is true, then (1) and (2) have to be true

Example

Show that $f(x) = x^2 - 2x + 1$ is a continuous function.

CONTINUOUS means continuous at all real numbers. Show that part (3) of definition is met for all real number *c*. By what we know about polynomials so far, we have

$$\lim_{x\to c} f(x) = f(c)$$

So

$$\lim_{x\to c}(x^2 - 2x + 1) = c^2 - 2c + 1$$

Part (3) is met and f(x) is continuous

Theorem 2.7.2

Polynomials are continuous functions.

Proof:

If P is polynomial and c is any real number then by theorem 2.5.2

$$\lim_{x\to c} p(x) = p(c)$$

Where *p* is a polynomial, and *c* is any real number. Since *c* is any real number, it follows that *p(x)* is continuous.

Example

Show that $f(x) = |x|$ is continuous

Rewrite f(x) as

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Show $\lim_{x\to c} f(x) = f(c)$ for any real number *c*.

Let $c \geq 0$. *Then* $f(c) = c$ *by definition of f (x).*

Also $\lim_{x\to c} f(x) = \lim_{x\to c}|x| = \lim_{x\to c} x = c$ Since $c \geq 0$

x may be negative to begin with, but since it approaches c which is positive or 0, we use the first part of the definition of f(x) to evaluate the limit
That is just f(x) = x which is a polynomial and hence we get the desired result.

Now let $c < 0$. Then again $f(c) = -c$ by definition of f (x) and

$$\lim_{x \to c} f(x) = \lim_{x \to c} |x| = \lim_{x \to c} x = -c$$

x may be Positive or 0 to begin with, but since it approaches *c* which is negative, we use the Second part of the definition of f(x) to evaluate the limit. That is just f(x) = -x which is a polynomial and hence we get the desired result.

Properties of Continuous Functions

Theorem 2.7.3

If the function f and g are continuous at c, then

a) f + g is continuous at c;

b) f – g is continuous at c;

c) f . g is continuous at c;

d) f/g is continuous at c if $g(c) \neq 0$ and is discontinuous at c if g(c)=0

PROOF

Let f and g be continuous function at the number c
Then

$$\lim_{x \to c} f(x) = f(c)$$

$$\lim_{x \to c} g(x) = g(c)$$

So

$$\lim_{x \to c} f(x).g(x) = \lim_{x \to c} f(x).\lim_{x \to c} g(x) \quad \text{by Limit Rules}$$

$$= f(c).g(c) \quad \text{by continuity of } f \text{ and } g$$

Continuity of Rational Functions

Example

Where is $h(x) = \dfrac{x^2 - 9}{x^2 - 5x + 6}$ continuous?

Since the top and the bottom functions in h are polynomials, they are continuous everywhere Hence, by property (d) of theorem 2.7.3, h will be continuous at all points c as long as $g(c) \neq 0$.

$$x^2 - 5x + 6 = 0$$

Will give us all the x values where h will be discontinuous. These are $x = 2 \quad x = 3$ which you get after solving the above equation for x.

Continuity of Composition of functions

Theorem 2.7.5

Let limit stand for one of the limits $\lim_{x \to c}$, $\lim_{x \to c^+}$, $\lim_{x \to c^-}$, $\lim_{x \to +\infty}$, or $\lim_{x \to -\infty}$. If $\lim g(x) = L$ and if the function f is continuous at L, Then $\lim f(g(x)) = f(L)$. that is $\lim f(g(x)) = f(\lim g(x))$.

Example

$$f(x) = \left|5 - x^2\right|$$

Here, $f(x) = |x|, \quad g(x) = 5 - x^2$

SO by theorem 2.7.5

$$\lim_{x \to 3}\left|5 - x^2\right| = \left|\lim_{x \to 3} 5 - x^2\right| = |-4| = 4$$

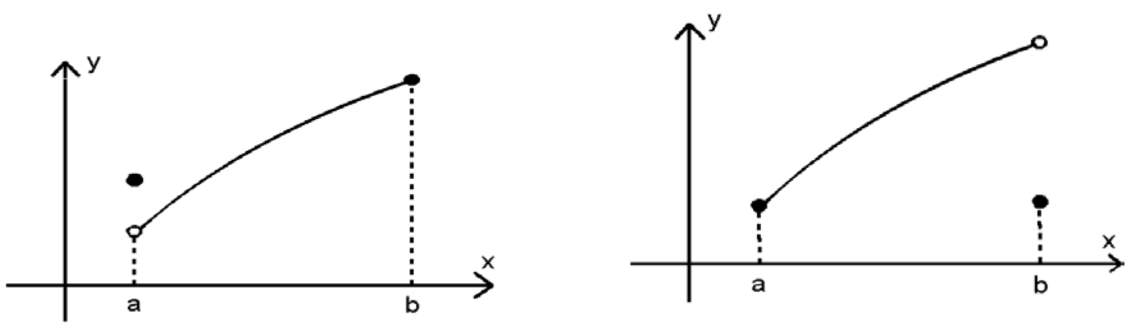
Theorem 2.7.6

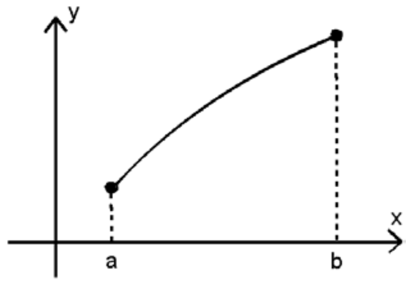
If the function g is continuous at the point c and the function f is continuous at the point g (c), then the composition $f \circ g$ is continuous at c.

Continuity from the left and right

Definition we use does not incorporate end points as at end points only left hand or right hand limits make sense

(a) (b)





(c)

- Graph of function in a) shows that *f* is discontinuous at *a*
- Graph of function in b) shows that *f* is discontinuous at *b*
- Graph of function in c) shows that *f* is continuous at *a* and *b*

Definition 2.7.7

A function f is called **continuous from the left at point c** if the conditions in the left column below are satisfied, and is called **continuous from the right at the point c** if the conditions in the right column are satisfied.

1. f(c) is defined.
2. $\lim\limits_{x \to c^-} f(x)$ *exists.*
3. $\lim\limits_{x \to c^-} f(x) = f(c).$

1`. f(c) is defined

2`. $\lim\limits_{x \to c^+} f(x)$ *exists.*

3`. $\lim\limits_{x \to c^+} f(x) = f(c).$

Definition 2.7.8

A function f is said to be continuous on a closed interval [a, b] if the following conditions are satisfied:

1. *f* is continuous on (a, b).
2. *f* is continuous from the right at a.
3. *f* is continuous from the left at b.

EXAMPLE

Show that $f(x) = \sqrt{9 - x^2}$ is continuous on the interval [3,-3].By definition 2.7.8 and theorem 2.5.1(e), for *c* in (3,-3)

$$\lim_{x \to c} f(x) = \lim_{x \to c} \sqrt{9 - x^2} = \sqrt{\lim_{x \to c}(9 - x^2)} = \sqrt{9 - c^2} = f(c)$$

So *f* is continuous on (3,-3) Also

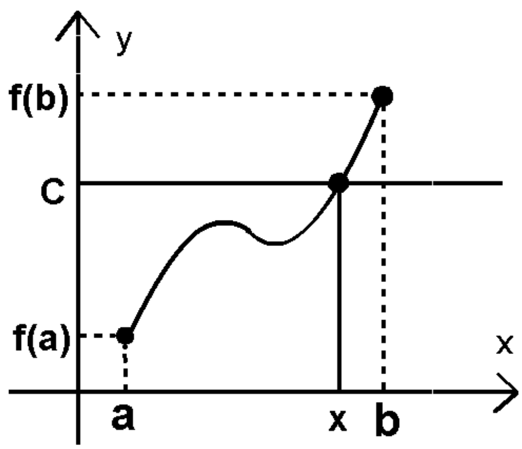
$$\lim_{x\to 3^-}\sqrt{9-x^2} = \sqrt{\lim_{x\to 3^-}(9-x^2)} = f(3) = 0$$

$$\lim_{x\to -3^+}\sqrt{9-x^2} = \sqrt{\lim_{x\to -3^+}(9-x^2)} = f(3) = 0$$

Why approach 3 from the left and –3 from the right? Well, draw the graph of this function and you will see WHY!?? So f is continuous on [-3, 3].

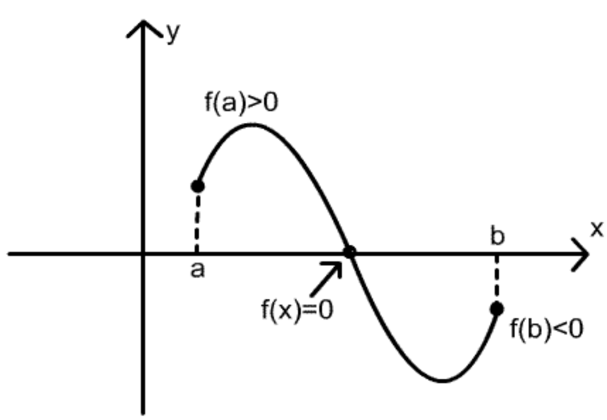
Intermediate Value Theorem(Theorem 2.7.9)

If *f* is continuous on a closed interval [a, b] and C is any number between *f(a)* and *f(b)*, inclusive, then there is at least one number *x* in the interval [a, b] such that *f(x)* = C.



Theorem 2.7.10

If *f* is continuous on [a, b], and if *f(a)* and *f(b)* have opposite signs, then there is at least one solution of the equation *f(x)* = 0 in the interval (a, b).



Example

$$x^3 - x - 1 = 0$$

Cannot be solved easily by factoring. However, by the MVT, f(1) = -1 and f(2) = 5 implies that the equation has one solution in the interval (1,2).