Lecture 11

Limits: A Rigorous Approach

In this section we will talk about

-**Formal Definition of Limit**

- Left-hand and Right-hand Limits

So far we have been talking about limits informally. We haven't given FORMAL mathematical definitions of limit yet. We will give a formal definition of a limit. It will include the idea of left hand and right hand limits. We intuitively said that

$$\lim_{x \to a} f(x) = L$$

means that as x approaches a, f (x) approaches L. The concept of "approaches" is intuitive.
The concept of "approaches" is intuitive so far, and does not use any of the concepts and theory of Real numbers we have been using so far.
So let's formalize LIMIT

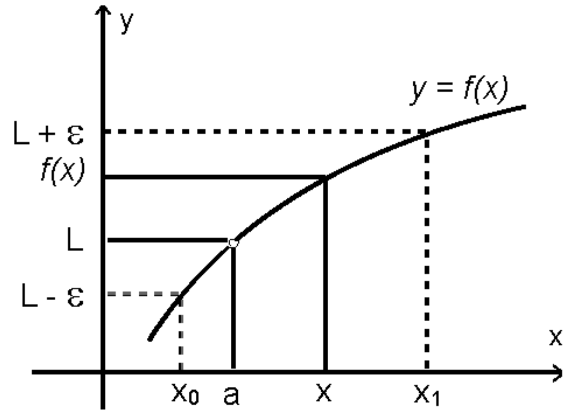
Note that when we talked about "f (x) approaches L" as "x approaches a" from left and right, we are saying that we want f(x) to get as close to L as we want provided we can get x as close to *a* as we want as well, but maybe not equal to a since f (*a*) maybe undefined and f (*a*) may not equal L. So naturally we see the idea of INTERVALS involved here.

I will rephrase the statement above in intervals as

$$\textit{For any number } \varepsilon > 0 \textit{ if we can find an open } \text{interval } (x_0, x_1)$$
$$\textit{on the } x-\textit{axis containing a point a such that}$$
$$L - \varepsilon < f(x) < L + \varepsilon$$
$$\textit{for each } x \textit{ in } (x_0, x_1) \textit{ except possibly } x = a. \textit{Then}$$
$$\textit{we say}$$
$$\lim_{x \to a} f(x) = L$$
$$\textit{So, } f(x) \textit{ is in the } \text{interval } (L - \varepsilon, L + \varepsilon)$$

Now you may ask, what is this ε all about??
Well, it is the number that signifies the idea of "f (x) being as close to L as we want to be" could be a very small positive number, and that why it Let's us get as close to f (x) as we want. Imagine it to be something like the number at the bottom is called a GOOGOLPLEX!!

$$\frac{1}{10^{10^{100}}}$$



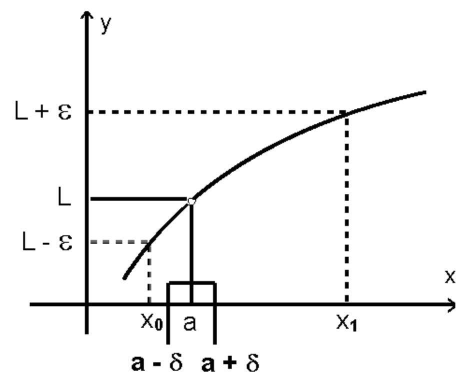
So $\varepsilon > 0$ but very close to it, and for ANY such ε we can find an interval on the x-axis that can confine *a*.
Let's pin down some details. Notice that
When we said

$L-\varepsilon < f(x) < L+\varepsilon$

holds for every x in the interval (x0, x1) (except possibly at x= a), it is the same as saying that the same inequality hold of all x in the interval set $(x_0, a) \cup (a, x_1)$.

But then the inequality $L-\varepsilon < f(x) < L+\varepsilon$ holds in any subset of this interval, namely $(x_0, a) \cup (a, x_1)$

ε is any positive real number smaller than $a - x_0$ and $x_1 - a$.
Look at figure below



$L-\varepsilon < f(x) < L+\varepsilon$

can be written as

$| f(x) - L | < \varepsilon$ *and* $(a-\delta, a) \cup (a,\ a+\delta)$ as $0 < | x-a | < \delta$.

Are same sets by picking numbers close to a and a-delta. Have them look at definition again and talk.
Let's use this definition to justify some GUESSES we made about limits in the previous lecture.

Example

Find

$$\lim_{x\to 2}(3x-5) = 1$$

Given any positive number ε we can find an δ such that

$$\left|(3x-5)-1\right| < \varepsilon \quad \text{if } x \text{ satisfies} \quad 0 < \left|x-2\right| < \delta$$

In this example we have

$f(x) = 3x - 5 \quad L = 1 \quad a = 2$

So our task is to find out the ε for which will work for any δ we can say the following

$$\left|(3x-5)-1\right| < \varepsilon \quad \text{if} \quad 0 < |x-2| < \delta$$

$$\Rightarrow |3x-6| < \varepsilon \quad \text{if} \quad 0 < |x-2| < \delta$$

$$\Rightarrow 3|x-2| < \varepsilon \quad \text{if} \quad 0 < |x-2| < \delta$$

$$\Rightarrow |x-2| < \frac{\varepsilon}{3} \quad \text{if} \quad 0 < |x-2| < \delta$$

Now we find our δ that makes our statement true. Note that the first part of the statement depends on the second part for being true. So our CHOICE of δ will determine the trueness of the first part.

I let $\frac{\varepsilon}{3}$ in the second part which makes the first part true. So we have

$$|x-2| < \frac{\varepsilon}{3} \quad \text{if} \quad 0 < |x-2| < \frac{\varepsilon}{3}$$

Hence we have proved that

$$\lim_{x \to 2}(3x-5) = 1$$

Example

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Show that $\lim_{x \to 0} f(x)$ does not exit.

Suppose that the Limit exists and its L. So $\lim_{x \to 0} f(x) = L$. Then for any $\varepsilon > 0$ we can find $\delta > 0$ such that

$$\left|f(x) - L\right| < \varepsilon \quad \text{if} \quad 0 < |x-0| < \delta$$

In particular, if we take $\varepsilon = 1$ there is a $\varepsilon > 0$ such that

$$\left|f(x) - L\right| < 1 \quad \text{if} \quad 0 < |x-0| < \delta$$

But $x = \frac{\delta}{2}$ and $x = -\frac{\delta}{2}$ both satisfy requirement above, so

$$\left| f(\frac{\delta}{2}) - L \right| < 1 \quad \text{and} \quad \left| f(-\frac{\delta}{2}) - L \right| < 1$$

But $\frac{\delta}{2}$ is positive and $-\frac{\delta}{2}$ is negative, so

$$f(\frac{\delta}{2}) = 1 \quad \text{and} \quad f(-\frac{\delta}{2}) = -1$$

So we get

$$|1 - L| < 1 \quad \text{and} \quad |-1 - L| < 1$$

$$\Rightarrow 0 < L < 2 \qquad \text{and} \qquad -2 < L < 0$$

But this is a contradiction since L cannot be between these two bounds at the same time.