Lecture 10

LIMITS AND COMPUTATIONAL TECHNIQUES

Previous lecture was about graphical view of Limits. This lecture will focus on algebraic techniques for finding Limits. Results will be intuitive again. Proofs will come later after we define **LIMIT** Mathematically. We will see how to use limits of basic functions to compute limits of complicated functions.
In this section, if I write down Lim x->a f(x), I will assume that f(x) will have a limit that matches from both sides and so the LIMIT EXISTS for f (x). So I won't distinguish between left and right hand limits.
We begin with a table of LIMITS of two basic functions
The functions are

$$f(x) = k$$
$$g(x) = x$$

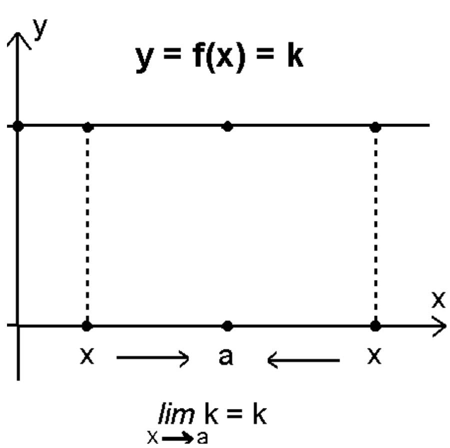
Here is the table of the limits and the same information from the graph

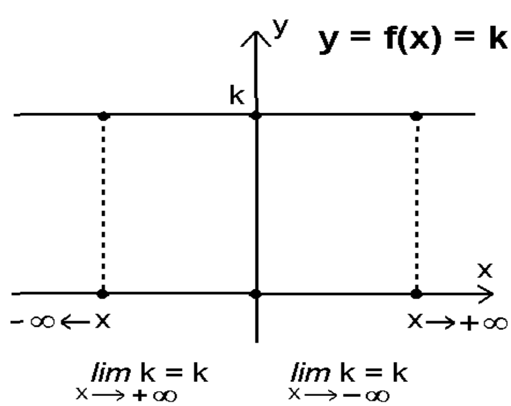
$f(x) = k$

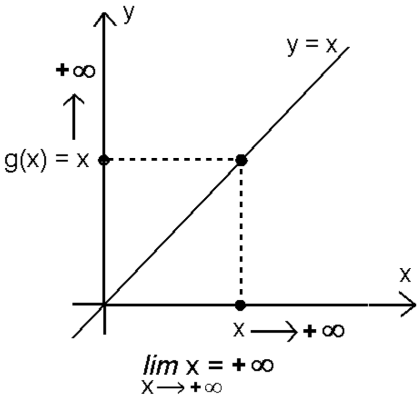
Limit	Example
$\lim_{x \to a} k = k$	$\lim_{x \to 2} 3 = 3, \quad \lim_{x \to -2} 3 = 3$
$\lim_{x \to +\infty} k = k$	$\lim_{x \to +\infty} 3 = 3, \quad \lim_{x \to +\infty} 0 = 0$
$\lim_{x \to -\infty} k = k$	$\lim_{x \to -\infty} 3 = 3, \quad \lim_{x \to -\infty} 0 = 0$

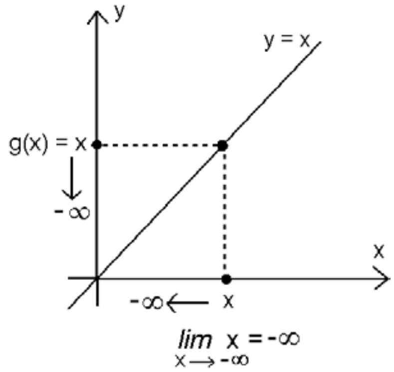
$g(x) = x$

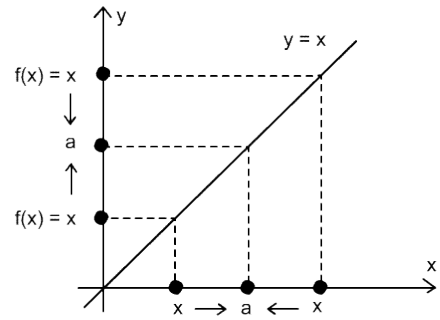
Limit
$\lim_{x \to a} x = a$
$\lim_{x \to +\infty} x = +\infty$
$\lim_{x \to -\infty} x = -\infty$











Here we have a theorem that will help with computing limits. Won't prove this theorem, but some of the parts of this theorem are proved in Appendix C of your text book.

THEOREM 2.5.1

Let Lim stand for one of the limits

$$\lim_{x\to a}, \quad \lim_{x\to a^-}, \quad \lim_{x\to a^+}, \quad \lim_{x\to +\infty}, \quad \lim_{x\to -\infty}$$

$$if\ L_1 = \lim f(x)\ and\ L_2 = \lim g(x)\ both\ exists,\ then$$

$$a)\ \lim[\ f(x) + g(x)] = \lim\ f(x) + \lim g(x)$$
$$= L_1 + L_2$$
$$b)\ \lim[\ f(x) - g(x)] = \lim\ f(x) - \lim g(x)$$
$$= L_1 - L_2$$
$$c)\ \lim[\ f(x)\,.\,g(x)] = \lim\ f(x)\,.\lim g(x)$$
$$= L_1.L_2$$
$$d)\ \lim[\frac{f(x)}{g(x)}\ ] = \frac{\lim\ f(x)}{\lim g(x)}$$
$$= \frac{L_1}{L_2}\quad (L_2 \neq 0)$$

For the Last theorem, say things like "Limit of the SUM is the SUM of the LIMITS etc.
Parts a) and c) of the theorem apply to as many functions as you want
Part a) gives

$$\lim[\,f_1(x) + f_2(x) + ... + f(x_n)]$$
$$= \lim\ f_1(x) + \lim\ f_2(x) + ... \lim\ f(x_n)$$

Part c) gives

$$\lim[f_1(x)\cdot f_2(x)\cdot\ldots\cdot f_n(x)]$$
$$=\lim f_1(x)\cdot\lim f_2(x)\cdot\ldots\cdot\lim f_n(x)$$

Also if

$$f_1 = f_2 = \ldots = f_n \text{ then } \lim[f(x)]^n = [\lim f(x)]^n$$

From this last result we can say that

$$\lim_{x\to a}(x^n) = [\lim_{x\to a} x^n] = a^n$$

This is a useful result and we can use it later.
Another useful result follows from part c) of the theorem. Let f(x) = k in part c), where k is a constant (number).

$$\lim[kg(x)] = \lim(k)\cdot\lim g(x) = k\cdot\lim g(x)$$

So a constant factor can be moved through a limit sign

LIMITS OF POLYNOMIAL
Polynomials are functions of the form

$$f(x) = b_n x^n + b_{n-1}x^{n-1} + \ldots. + b_1 x + b_0$$

Where the a's are all real numbers Let's find the Limits of polynomials and x approaches a numbers *a*

Example

$$\lim_{x\to 5}(x^2 - 4x + 3)$$
$$= \lim_{x\to 5} x^2 - \lim_{x\to 5} 4x + \lim_{x\to 5} 3$$
$$= \lim_{x\to 5} x^2 - 4\lim_{x\to 5} x + \lim_{x\to 5} 3$$
$$= (5)^2 - 4(5) + 3 = 8$$

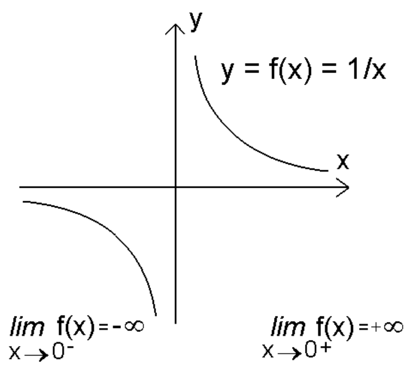
Theorem 2.5.2

Proof

$$\lim_{x\to a} p(x) = \lim_{x\to a}(c_0 + c_1 x + \ldots + c_n x^n)$$
$$= \lim_{x\to a} c_0 + \lim_{x\to a} c_1 x + \ldots + \lim_{x\to a} c_n x^n$$
$$= \lim_{x\to a} c_0 + c_1 \lim_{x\to a} x + \ldots + c_n \lim_{x\to a} x^n$$
$$= c_0 + c_1 a + \ldots + c_n a^n = p(a)$$

Limits Involving $\frac{1}{x}$

Let's look at the graph of $f(x) = \frac{1}{x}$



Then by looking at the graph AND by looking at the TABLE of values we get the following Results

	Values	Conclusion
x 1/x	1 .. .01 .. .001 ... 1 .. 100 .. 1000 ...	$x \to 0^+$ $1/x \to +\infty$
x 1/x	-1 .. -.01 .. -.001 ... -1 .. -100 .. -1000 ...	$x \to 0^-$ $1/x \to -\infty$
x 1/x	1 .. 100 .. 1000 ... 1 .. .01 .. .001 ...	$x \to +\infty$ 1/x decreases towards 0
x 1/x	-1 .. -100 .. -1000 ... -1 .. -.01 .. -.001 ...	$x \to -\infty$ 1/x increases towards 0

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty$$

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

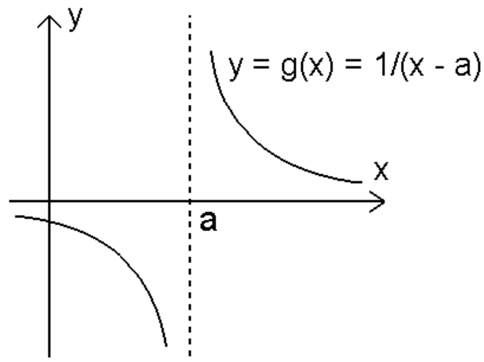
$$\lim_{x \to +\infty} \frac{1}{x} = 0$$

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

For every real number *a,* the function

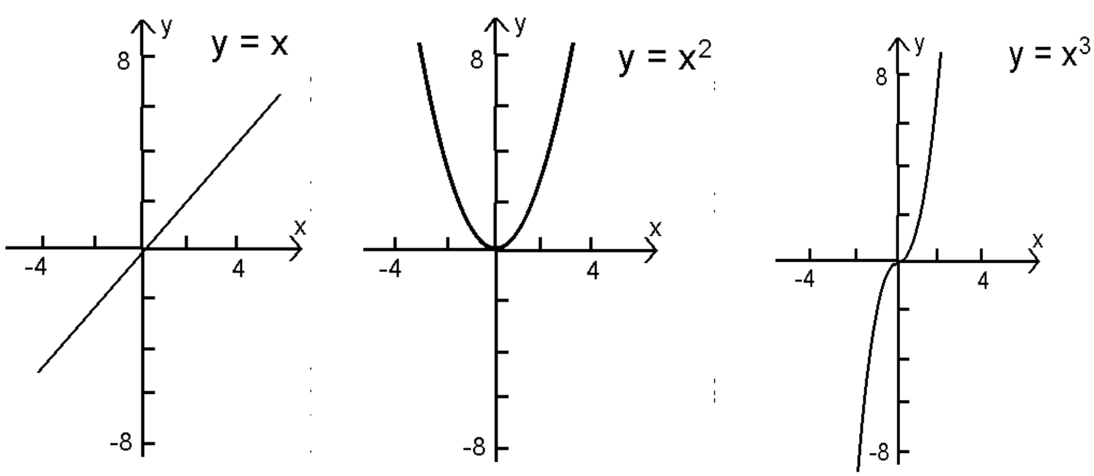
$g(x) = \frac{1}{x-a}$ is a translation of $f(x) = \frac{1}{x}$.

So we an say the following about this function



LIMITS OF POLYNOMIALS AS X GOES TO +INF AND –INF

From the graphs given here we can say the following about polynomials of the form



$$\lim_{x\to+\infty} x^n = +\infty \quad n = 1, 2, 3, \ldots$$

$$\lim_{x\to-\infty} x^n = \begin{cases} +\infty & n = 2, 4, 6, \ldots \\ -\infty & n = 1, 3, 5, \ldots \end{cases}$$

EXAMPLE

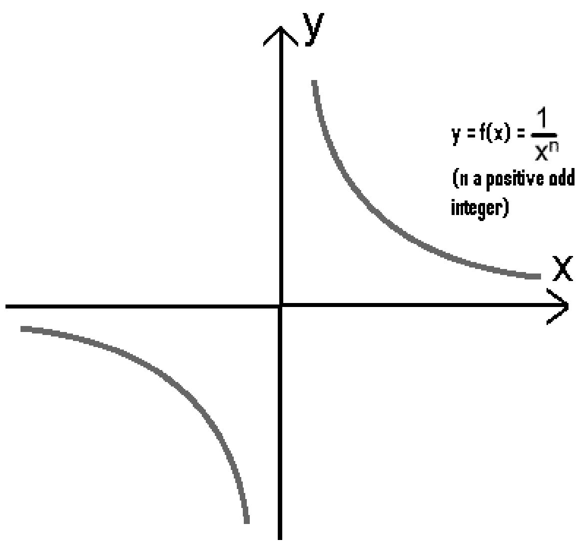
$$\lim_{x\to+\infty} 2x^5 = +\infty$$

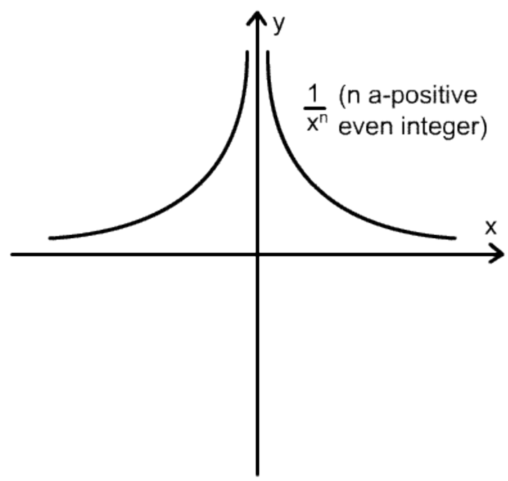
$$\lim_{x\to+\infty} -7x^6 = -\infty$$

$$\lim_{x\to+\infty} \frac{1}{x^n} = \left(\lim_{x\to+\infty} \frac{1}{x^n}\right)^n = 0$$

$$\lim_{x\to-\infty} \frac{1}{x^n} = \left(\lim_{x\to-\infty} \frac{1}{x^n}\right)^n = 0$$

HERE are the graphs of the functions





$y = f(x) = 1/x^n$ (*n is positive* integer)

Limit as x goes to +inf or –inf of a polynomial is like the Limits of the highest power of x

$$\lim_{x\to+\infty} (c_0 + c_1 x + \ldots + c_n x^n) = \lim_{x\to+\infty} c_n x^n$$

Motivation

$$(c_0 + c_1 x + ... + c_n x^n) = x^n (\frac{c_0}{x^n} + \frac{c_1}{x^{n-1}} + ... + c_n)$$

Factor out x^n , and then from what we just saw about the limit of $\frac{1}{x^n}$, everything goes to 0 as $x \to +\infty$ or $x \to -\infty$ except c_n

Limits of Rational Functions as x goes to *a*

A rational function is a function defined by the ratio of two polynomials

Example

Find $\lim_{x\to 2} \frac{5x^3 + 4}{x - 3}$

Solution:

$$= \frac{\lim_{x\to 2} 5x^3 + 4}{\lim_{x\to 2} x - 3}$$

$$= \frac{5(2)^3 + 4}{2 - 3} = -44$$

We used d) of theorem 2.5.1 to evaluate this limit. We would not be able to use it if the denominator turned out to be 0 as that is not allowed in Mathematics. If both top and bottom approach 0 as x approaches a, then the top and bottom will have a common factor of x – a. In this case the factors can be cancelled and the limit works out.

Example

$$\lim_{x\to 2} \frac{x^2 - 4}{x - 2}$$

$$= \lim_{x\to 2} \frac{(x+2)(x-2)}{(x-2)}$$

$$= \lim_{x\to 2} (x+2) = 4$$

Note that x is not equal to two after Simplification for the two functions to be the same. Nonetheless, we calculated the limit as if we were substituting x = 2 using rule for polynomials That's ok since REALLY LIMIT means you are getting close to 2, but not equaling it!!

What happens if in a rational functions, the bottom limit is 0, but top is not?? It's like the limit as x goes to 0 of f(x) = 1/x.

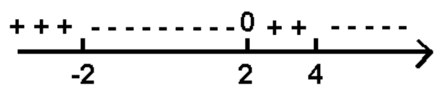
- The limit may be + inf
- The limits may be –inf
- +inf from one side and –inf from another

Example

Find $\lim_{x \to 4^+} \frac{2-x}{(x-4)(x+2)}$

The top is –2 as x goes to 4 from right side. The bottom goes to 0, so the limit will be inf of some type. To get the sign on inf, Let's analyze the sign of the bottom for various values of real numbers

Break the number line into 4 intervals as in



The important numbers are the ones that make the top and bottom zero. As x approaches 4 from the right, the ratio stays negative and the result is –inf. You can say something about what happens from the left. Check yourselves by looking at the pic.

So $\lim_{x \to 4^+} \frac{2-x}{(x-4)(x+2)} = -\infty$

LIMITS of Rational Functions as x goes to +inf and -inf

Algebraic manipulations simplify finding limits in rational functions involving +inf and –inf.

Example

$\lim_{x \to -\infty} \frac{4x^2 - x}{2x^3 - 5}$

Divide the top and the bottom by the highest power of x

$$\lim_{x\to-\infty}\frac{\frac{4x^2}{x^3}-\frac{x}{x^3}}{\frac{2x^3}{x^3}-\frac{5}{x^3}}=\lim_{x\to-\infty}\frac{\frac{4}{x}-\frac{1}{x^2}}{2-\frac{5}{x^3}}=\frac{\lim\limits_{x\to-\infty}(\frac{4}{x}-\frac{1}{x^2})}{\lim\limits_{x\to-\infty}(2-\frac{5}{x^3})}$$

$$=\frac{4\lim\limits_{x\to-\infty}\frac{1}{x}-\lim\limits_{x\to-\infty}\frac{1}{x^2}}{2-5\lim\limits_{x\to-\infty}\frac{1}{x^3}}$$

$$=\frac{4(0)-0}{2-5(0)}=0$$

Quick Rule for finding Limits of Rational Functions as x goes to +inf or –inf

$$\lim_{x\to+\infty}\frac{c_0+c_1x+...+c_nx^n}{d_0+d_1x+...+d_nx^n}=\lim_{x\to+\infty}\frac{c_nx^n}{d_nx^n}$$

$$\lim_{x\to-\infty}\frac{c_0+c_1x+...+c_nx^n}{d_0+d_1x+...+d_nx^n}=\lim_{x\to-\infty}\frac{c_nx^n}{d_nx^n}$$

Not true if x goes to a finite number *a*.

Example

$$\lim_{x\to-\infty}\frac{4x^2-x}{2x^3-5}=\lim_{x\to-\infty}\frac{4x^2}{2x^3}=\lim_{x\to-\infty}\frac{2}{x}=0$$

Same answer as the one we got earlier from algebraic manipulations

