Lecture 1

Coordinates, Graphs and Lines

What is Calculus??

Well, it is the study of the continuous rates of the change of quantities. It is the study of how various quantities change with respect to other quantities. For example, one would like to know how distance changes with respect to (from now onwards we will use the abbreviation w.r.t) time, or how time changes w.r.t speed, or how water flow changes w.r.t time etc. You want to know how this happens continuously. We will see what continuously means as well.

In this lecture, we will talk about the following topics:

-Real Numbers

-Set Theory

-Intervals

-Inequalities

-Order Properties of Real Numbers

Let's start talking about Real Numbers. We will not talk about the COMPLEX or IMANGINARY numbers, although your text has something about them which you can read on your own. We will go through the history of REAL numbers and how they popped into the realm of human intellect. We will look at the various types of REALS - as we will now call them. So Let's START.

The simplest numbers are the ***natural numbers***

<u>Natural Numbers</u>

$1, 2, 3, 4, 5, \ldots$

They are called the natural numbers because they are the first to have crossed paths with human intellect. Think about it: these are the numbers we count things with. So our ancestors used these numbers first to count, and they came to us naturally! Hence the name

NATURAL!!!

The natural numbers form a subset of a larger class of numbers called the ***integers.*** I have used the word SUBSET. From now onwards we will just think of SET as a COLLECTION OF THINGS.

This could be a collection of oranges, apples, cars, or politicians. For example, if I have the SET of politicians then a SUBSET will be just a part of the COLLECTION. In mathematical notation we say A is subset of B if $\forall x \in A \Rightarrow x \in B$.Then we write $A \subseteq B$.

Set

The collection of well defined objects is called a set. For example

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Subset

A portion of a set B is a subset of A iff every member of B is a member of A. e.g. one subset of above set is

{George Bush, Tony Blair}

The curly brackets are always used for denoting SETS. We will get into the basic notations and ideas of sets later. Going back to the Integers. These are

$$...,-4,-3,-2,-1,0,1,2,3,4,...$$

So these are just the natural numbers, plus a 0, and the NEGATIVES of the natural numbers.

The reason we didn't have 0 in the natural numbers is that this number itself has an interesting story, from being labeled as the concept of the DEVIL in ancient Greece, to being easily accepted in the Indian philosophy, to being promoted in the use of commerce and science by the Arabs and the Europeans. But here, we accept it with an open heart into the SET of INTEGERS.

What about these NEGATIVE Naturals??? Well, they are an artificial construction. They also have a history of their own. For a long time, they would creep up in the solutions of simple equations like

$x+2=0$. The solution is $x=-2$

So now we have the Integers plus the naturals giving us things we will call REAL numbers. But that's not all. There is more. The integers in turn are a subset of a still larger class of numbers called the **rational numbers**. With the exception that division by zero is ruled out, the rational numbers are formed by taking ratios of integers.

Examples are

$$\frac{2}{3},\frac{7}{5},\frac{6}{1},\frac{-5}{2}$$

Observe that every integer is also a rational number because an integer p can be written as a ratio. So every integer is also a rational. Why not divide by 0? Well here is why:

If x is different from zero, this equation is contradictory; and if x is equal to zero, this equation is satisfied by any number y, so the ratio does not have a unique value a situation that is mathematically unsatisfactory.

$$x/0=y \Rightarrow x=0.y \Rightarrow x=0$$

For these reasons such symbols are not assigned a value; they are **said to be undefined.**

So we have some logical inconsistencies that we would like to avoid. I hope you see that!! Hence, no division by 0 allowed! Now we come to a very interesting story in the history of the development of Real numbers. The discovery of IRRATIONAL numbers.

Pythagoras was an ancient Greek philosopher and mathematician. He studied the properties of numbers for its own sake, not necessarily for any applied problems. This was a major change in mathematical thinking as math now took on a personality of its own. Now Pythagoras got carried away a little, and developed an almost religious thought based on math. He concluded that the size of a physical quantity must consist of a certain whole number of units plus some fraction m / n of an additional unit. Now rational numbers have a unique property that if you convert them to decimal notation, the numbers following the decimal either end quickly, or repeat in a pattern forever.

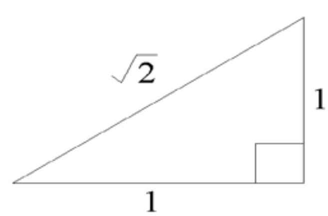
Example:

$$\frac{1}{2} = 0.500000\ldots = 0.5$$

$$\frac{1}{3} = 0.33333\ldots$$

This fit in well with Pythagoras' beliefs. All is well. But this idea was shattered in the fifth century B.C. by Hippasus of Metapontum who demonstrated the existence of *irrational numbers,* that is, numbers that cannot be expressed as the ratio of integers.

Using geometric methods, he showed that the hypotenuse of the right triangle with base and opposite side equal to 1 cannot be expressed as the ratio of integers, thereby proving that $\sqrt{2}$ is an IRRATIONAL number. The hypotenuse of this right triangle can be expressed as the ratio of integers.

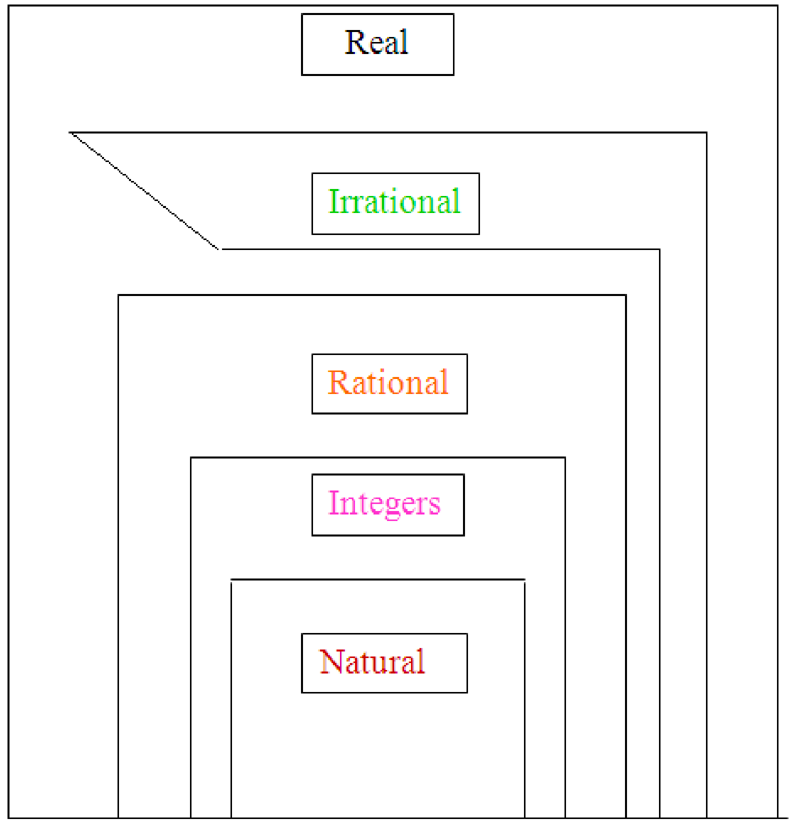


Other examples of irrational numbers are

$$Cos\,19^0, 1+\sqrt{2}$$

The rational and irrational numbers together comprise a larger class of numbers, called REAL NUMBERS *or* sometimes the REAL NUMBER SYSTEM. **So here is a pictorial summary of the hierarchy of REAL NUMBNERS.**

Pictorial summary of the hierarchy of REAL NUMBNERS

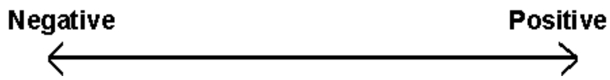


COORDINATE Line

In the 1600's, *analytic geometry* was "developed". It gave a way of describing algebraic formulas by geometric curves and, conversely, geometric curves by algebraic formulas. So basically you could DRAW PICTURES OF THE EQUATIONS YOU WOULD COME ACROSS, AND WRITE DOWN EQUATIONS OF THE PICTURES YOU RAN INTO!

The developer of this idea was the French mathematician, Descartes .The story goes that he wanted to find out as to what Made humans HUMANS?? Well, he is said to have seated himself in a 17th century furnace (it was not burning at the time!) and cut himself from the rest of the world. In this world of cold and darkness, he felt all his senses useless. But he could still think!!!! So he concluded that his ability to think is what made him human, and then he uttered the famous line: " I THINK, THEREFORE I AM" .In analytic geometry , the key step is to establish a correspondence between real numbers and points on a line. We do this by arbitrarily designating one of the two directions along the line as the positive direction and the other as the negative direction.

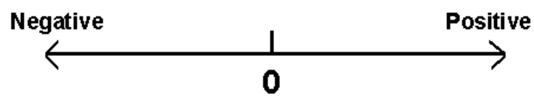
So we draw a line, and call the RIGHT HAND SIDE as POSITIVE DIRECTION, and the LEFT HAND SIDE as NEGATIVE DIRECTION. We could have done it the other way around too. But, since what we just did is a cultural phenomenon where right is $+$ and left is $-$, we do it this way. Moreover, this has now become



a standard in doing math, so anything else will be awkward to deal with. The positive direction is usually marked with an arrowhead so we do that too. Then we choose an arbitrary point and take that as our point of reference.

We call this the ORIGIN, and mark it with the number 0. So we have made our first correspondence between a real number and a point on the Line. Now we choose a unit of measurement, say 1 cm. It can be anything really. We use this unit of measurement to mark of the rest of the numbers on the line. Now this line, the origin, the positive direction, and the unit of measurement define what is called a coordinate line or sometimes a real line.

With each real number we can now associate a point on the line as fo1lows:



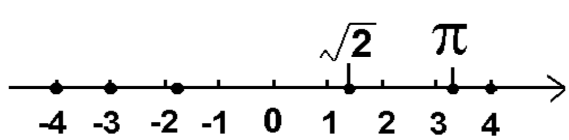
- Associate the origin with the number 0.
- Associate with each positive number r the point that is a distance of r units (this is the unit we chose, say 1 cm) in the positive direction from the origin.
- Associate with each negative number $-r$ the point that is a distance of r units in the negative direction from the origin.

The real number corresponding to a point on the line is called the coordinate of the point.

Example 1:

In Figure we have marked the locations of the points with coordinates $-4, -3, -1.75, -0.5, \pi, \sqrt{2}$ and 4. The locations of π and $\sqrt{2}$ which are approximate, were obtained from their decimal approximations,

$\pi = 3.14$ and $\sqrt{2} = 1.41$



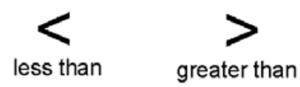
It is evident from the way in which real numbers and points on a coordinate line are related that each real number corresponds to a single point and each point corresponds to a single real number. To describe this fact we say that the real numbers and the points on a coordinate line are in one-to-one correspondence.

Order Properties

In mathematics, there is an idea of ORDER of a SET. We won't go into the general concept, since that involves SET THEORY and other high level stuff. But we will define the ORDER of the real number set as follows:

For any two real numbers a and b, if $b-a$ is positive, then we say that $b>a$ or that $a<b$.

Here I will assume that we are all comfortable working with the symbol "$<$" which is read as "less than" and the symbol "$>$" which is read as "greater than." I am assuming this because this stuff was covered in algebra before Calculus. So with this in mind we can write the above statement as



If $b-a$ is positive, then we say that $b>a$ or that $a<b$. A statement involving $<$ or $>$ are called an INEQUALITY. Note that the inequality $a<b$ can also be expressed as $b>a$.

So ORDER of the real number set in a sense defines the SIZE of a real number relative to another real number in the set. The SIZE of a real number *a* makes sense only when it is compared with another real *b*. So **the ORDER tells you how to "ORDER" the numbers in the SET and also on the COORDINATE LINE!**

A little more about inequalities. The inequality $a \le b$ is defined to mean that either $a<b$ or $a=b$.

So there are two conditions here. For example, the inequality $2 \le 6$ would be read as 2 is less than or it is equal to 6. We know that it's less than 6, so the inequality is true. SO IF ONE OF THE CONDITIONS IS TRUE, THEN THE INEQUALITY WILL BE TRUE, We can say a similar thing about. The expression $a<b<c$ is defined to mean that $a<b$ and $b<c$. It is also read as "b is between a and c".

As one moves along the coordinate line in the positive direction, the real numbers increase in size. In other words, the real numbers are ordered in an ascending manner on the number line, just as they are in the SET of REAL NUMBERS. So that on a horizontal coordinate line the inequality $a<b$ implies that a is to the left of b, and the inequality $a<b<c$ implies that a is to the left of b and b is to the left of c.

The symbol $a<b<c$ means $a<b$ and $b<c$. I will leave it to the reader to deduce the meanings of such symbols as $\le$ and $\ge$.

Here is an example of INEQUALITIES.

$a \le b < c$

$a \le b \le c$

$a < b < c < d$

Example:

Correct Inequalities

$3 < 8, \quad -7 < 1.5, \quad -12 \le x, \quad 5 \le 5$
$0 \le 2 \le 4, \quad 8 \ge 3, \quad 1.5 > -7$
$-\pi > -12, \quad 5 \ge 5, \quad 3 > 0 > -1 > -3$

Some incorrect inequalities are:

$2 \ge 4, \quad \pi \le 0, \quad 5 < -3$

REMARK: To distinguish verbally between numbers that satisfy $a \ge 0$ and those that satisfy $a > 0$, we shall call *a nonnegative* if $a \ge 0$ and *positive* if $a > 0$.

Thus, a nonnegative number is either positive or zero.

The following properties of inequalities are frequently used in calculus. We omit the proofs, but will look at some examples that will make the point.

THEOREM 1.1.1

$a)$ *If* $a < b$ *and* $b < c$, *then* $a < c$

$b)$ *If* $a < b$ *and* $a + c < b + c$, *then* $a - c < b - c$

$c)$ *If* $a < b$ *and* $ac < bc$, *when* c *is positive*
and $ac > bc$ *when* c *is negative.*

$d)$ *If* $a < b$ *and* $c < d$, *then* $a + c < b + d$

$e)$ *If* a *and* b *are both positive or both negative*
and $a < b$ *then* $\frac{1}{a} > \frac{1}{b}$

REMARK These five properties remain true if $<$ and $>$ are replaced by $\le$ and $\ge$

INTERVALS

We saw a bit about sets earlier. Now we shall assume in this text that you are familiar with the concept of a set and fully understand the meaning of the following symbols. However, we will give a short explanation of each.

Given two sets A and B

$a \in A$: *a is an element of the set A,*

$2 \in \{1, 2, 3, 4\}$

$a \notin A$: *a is NOT an element of the set A*

$5 \notin \{1, 2, 3, 4\}$

$\varnothing$ represents the Empty set, or the set that contains nothing.

$A \cup B$ represents the SET of all the elements of the Set A and the Set B taken together.

Example:

$A = \{1,2,3,4\}$, $B = \{1,2,3,4,5,6,7\}$, *then,* $A \cup B = \{1,2,3,4,5,6,7\}$

$A \cap B$ represents the SET of all those elements that are in Set A AND in Set B.

Example:

$A = \{1,2,3,4\}$, $B = \{1,2,3,4,5,6,7\}$, *then* $A \cap B = \{1,2,3,4\}$

$A = B$ means the A is exactly the same set as B

Example:

A = {1,2,3,4} and B = {1,2,3,4}, then A = C

and $A \subset B$ means that the Set A is contained in the Set B. Recall the example we did of the Set of all politicians!

{George Bush, Tony Blair} $\subset$ {George Bush, Toney Blair, Ronald Reagoan}

One way to specify the idea of a set is to list its members between braces. Thus, the set of all positive integers less than 5 can be written as

{ 1, 2, 3, 4}

and the set of all positive even integers can be written as

{ 2, 4, 6, …}

where the dots are used to indicate that only some of the members are explicitly and the rest can be obtained by continuing the pattern. So here the pattern is that the set consists of the even numbers, and the next element must be 8, then 10, and then so on. When it is inconvenient or impossible to list the members of a set, as would be if the set is infinite, then one can use the set- builder notation. This is written as

{ x : __________ }

which is read as **"the set of all *x* such that ______"** , In place of the line, one would state a property that specifies the set, Thus,

{ x : x is real number and 2 < x < 3}

is read**, "the set of all *x* such that *x* is a real number and 2 < *x* < 3,"** Now we know by now that
2 < *x* < 3 means that all the *x* between 2 and 3.

This specifies the "description of the elements of the set" This notation describes the set, without actually writing down all its elements.
When it is clear that the members of a set are real numbers, we will omit the reference to this fact. So we will write the above set as Intervals.

We have had a short introduction of Sets. Now we look particular kind of sets that play a crucial role in Calculus and higher math. These sets are sets of real numbers called ***intervals.*** What is an interval?

$$\{ x: 2< x< 3\}$$

Well, *geometrically*, an interval is a line segment on the co-ordinate line. S if *a* and *b* are real numbers such that $a < b$, then an interval will be just the line segment joining a and b.

But if things were only this simple! Intervals are of various types. For example, the question might be raised whether *a* and *b* are part of the interval? Or if *a* is, but *b* is not?? Or maybe both are?

Well, this is where we have to be technical and define the following:

The closed interval from a to b is denoted by [a, b] and is defined as

$$[a,b]=\{x:a\le x\le b\}$$

Geometrically this is the line segment

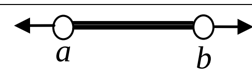
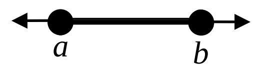
So this includes the numbers *a* and *b, a* and *b a* are called the END- POINTS of the interval.

The *open* interval from *a* to *b* is denoted by and is defined by

$$(a,b)=\{x:a<x<b\}$$

This excludes the numbers a and b. The square brackets indicate that the end points are included in the interval and the parentheses indicate that they are not.

Here are various sorts of intervals that one finds in mathematics. In this picture, the geometric pictures use solid dots to denote endpoints that are included in the interval and open dots to denote endpoints that are not.

INTERVAL NOTATION	SET NOTATION	GEOMETRIC PICTURE	CLASSIFICATION
(a,b)	$\{x : a < x < b\}$		Finite ; Open
$[a,b]$	$\{x : a \leq x \leq b\}$		Finite ; Closed
$[a,b)$	$\{x : a \leq x < b\}$		Finite ; Half-open
$(a,b]$	$\{x : a < x \leq b\}$		Finite ; Half-open
$(-\infty,b]$	$\{x : x \leq b\}$		Infinite ; Closed
$(-\infty,b)$	$\{x : x < b\}$		Infinite ; Open

As shown in the table, an interval can extend indefinitely in either the positive direction, the negative direction, or both. The symbols $-\infty$ (read "negative infinity") and $+\infty$ (read, 'positive infinity') do not represent numbers: the $+\infty$ indicates that the interval extends indefinitely in the positive direction, and the $-\infty$ indicates that it extends indefinitely in the negative direction.

An interval that goes on forever in either the positive or the negative directions, or both, on the coordinate line or in the set of real numbers is called an INFINITE interval. Such intervals have the symbol for infinity at either end points or both, as is shown in the table

$[a, +\infty)$	$\{x : x \geq a\}$		Infinite; closed
$(a, +\infty)$	$\{x : x > a\}$		Infinite; open
$(-\infty, +\infty)$	{ x:x is a real number}		Infinite; open and closed

An interval that has finite real numbers as end points are called finite intervals.

A finite interval that includes one endpoint but not the other is called *half-open* (or sometimes *half-closed).*

$[a, +\infty), (a, -\infty), (-\infty, b], (-\infty, b)$

Infinite intervals of the form $[a, +\infty)$ and $(-\infty, b]$ are considered to be closed because they contain their endpoint. Those of the form $(a, -\infty)$ and $(-\infty, b)$ has no endpoints; it is regarded to be both open and closed. As one of my Topology Instructors used to say:

"A set is not a DOOR! It can be OPEN, it can be CLOSED, and it can be OPEN and CLOSED!!

Let's remember this fact for good!" Let's look at the picture again for a few moments and digest the information. PAUSE 10 seconds.

SOLVING INEQUALITIES

We have talked about Inequalities before. Let's talk some more. First Let's look at an inequality involving and unknown quantity, namely *x*. Here is one: $x < 5$, $x = 1$, is a solution of this inequality as 1 makes it true, but $x = 7$ is not. So the set of all solutions of an inequality is called its solution set. The solution set of $x < 5$ will be

It is a fact, though we wont prove this that if one does not multiply both sides of an inequality by zero or an expression involving an unknown, then the operations in Theorem 1.1.1 will not change the solution set of the inequality. The process of finding the solution set of an inequality is called *solving* the Inequality.

Let's do some fun stuff, like some concrete example to make things a bit more focused

Example 4.

Solve $3+7x \le 2x-9$

Solution.

We shall use the operations of Theorem 1.1.1 to isolate *x* on one side of the inequality

$7x \le 2x-12$ *Subtracting* 3 *from both sides*

$5x \le -12$ *Subtracting* $2x$ *from both sides*

$x \le -12/5$ *Dividing both sides by* 5

Because we have not multiplied by any expressions involving the unknown *x,* the last inequality has the same solution set as the first. Thus, the solution set is the interval shown in Figure 1.1.6.

Example

Solve $7 \le 2-5x < 9$

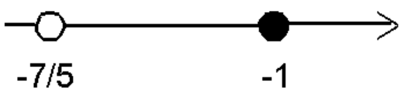
Solution ; The given inequality is actually a combination of the two inequalities

$$7 \le 2-5x \text{ and } 2-5x < 9$$

We could solve the two inequalities separately, then determine the value of x that satisfy both by taking the intersection of the solution sets , however, it is possible to work with the combined inequality in this problem

$$5 \le -5x < 7 \quad \textit{Subtracting 2 from both sides}$$
$$-1 \ge x > -7/5 \quad \textit{Dividing by } -5 \textit{ inequality symbols reversed}$$
$$-7/5 < x \le -1 \quad \text{Re}\textit{writing with smaller number first}$$
$$(-7/5, -1]$$

Thus the solution set is interval shown the figure



Example

Similarly, you can find