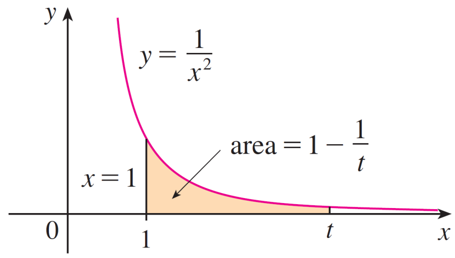
Improper Integrals

Type 1: Infinite Intervals

Consider the infinite region S that lies under the curve $y = 1/x^2$, above the x-axis, and to the right of the line $x = 1$. You might think that, since S is infinite in extend, its area must be infinite. However, this is not true. In fact, the area of the part of S that lies to the left of the line $x = t$ is

$$A(t) = \int_1^t \frac{1}{x^2}dx = \int_1^t x^{-2}dx = \frac{x^{-2+1}}{-2+1}\Bigg]_1^t = \frac{x^{-1}}{-1}\Bigg]_1^t = -\frac{1}{x}\Bigg]_1^t = 1 - \frac{1}{t}$$

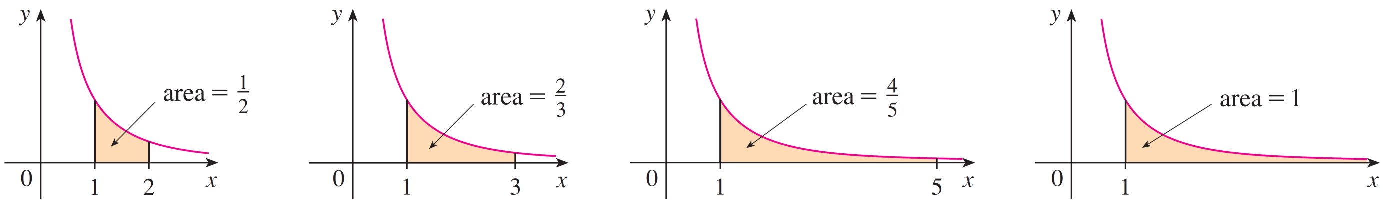


Notice that $A(t) < 1$ no matter how large t is chosen. Moreover, since

$$\lim_{t\to\infty} A(t) = \lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = 1$$

we can say that the area of the infinite region S is equal to 1 and we write

$$\int_1^\infty \frac{1}{x^2}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}dx = 1$$



DEFINITION OF AN IMPROPER INTEGRAL OF TYPE 1:

(a) If $\int_a^t f(x)dx$ exists for every number $t \geq a$, then $\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx$ provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)dx$ exists for every number $t \leq b$, then $\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty}\int_t^b f(x)dx$ provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) The improper integral $\int_{-\infty}^\infty f(x)dx$ is defined as $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$, where a is any real number. It is said to converge if both terms converge and diverge if either term diverges.

EXAMPLES:

1. Evaluate $\int\limits_1^\infty \frac{1}{x}dx$ if possible.

Solution: We have

$$\int\limits_1^\infty \frac{1}{x}dx = \lim_{t\to\infty}\int\limits_1^t \frac{1}{x}dx = \lim_{t\to\infty} \ln|x|]_1^t = \lim_{t\to\infty}(\ln t - \ln 1) = \lim_{t\to\infty}\ln t = \infty$$

The limit does not exist as a finite number and so the improper integral $\int\limits_1^\infty \frac{1}{x}dx$ is divergent.

2. Evaluate $\int\limits_2^\infty \frac{1}{x^2}dx$ if possible.

Solution: We have

$$\int\limits_2^\infty \frac{1}{x^2}dx = \lim_{t\to\infty}\int\limits_2^t \frac{1}{x^2}dx = \lim_{t\to\infty} -\frac{1}{x}\Big]_2^t = \lim_{t\to\infty}\left(-\frac{1}{t}+\frac{1}{2}\right) = 0+\frac{1}{2} = \frac{1}{2} \quad \text{(convergent)}$$

3. Evaluate $\int\limits_4^\infty \frac{1}{\sqrt{x}}dx$ if possible.

Solution: We have

$$\int\limits_4^\infty \frac{1}{\sqrt{x}}dx = \lim_{t\to\infty}\int\limits_4^t \frac{1}{\sqrt{x}}dx = \lim_{t\to\infty}\int\limits_4^t x^{-1/2}dx = \lim_{t\to\infty}\frac{x^{-1/2+1}}{-1/2+1}\Big]_4^t = \lim_{t\to\infty} 2\sqrt{x}]_4^t = \lim_{t\to\infty}\left(2\sqrt{t}-2\sqrt{4}\right) = \infty$$

The limit does not exist as a finite number and so the improper integral $\int\limits_4^\infty \frac{1}{\sqrt{x}}dx$ is divergent.

4. For what values of p is $\int\limits_1^\infty \frac{1}{x^p}dx$ convergent?

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Solution: We know that if $p=1$, then the integral is divergent, so let's assume that $p \neq 1$. Then

$$\int_1^\infty \frac{1}{x^p}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^p}dx = \lim_{t\to\infty}\int_1^t x^{-p}dx = \lim_{t\to\infty}\frac{x^{-p+1}}{-p+1}\bigg]_1^t = \lim_{t\to\infty}\frac{1}{(1-p)x^{p-1}}\bigg]_1^t = \lim_{t\to\infty}\frac{1}{1-p}\left(\frac{1}{t^{p-1}}-1\right)$$

If $p>1$, then $p-1>0$, so as $t\to\infty$, $t^{p-1}\to\infty$ and $\dfrac{1}{t^{p-1}}\to 0$. Therefore

$$\int_1^\infty \frac{1}{x^p}dx = \frac{1}{p-1} \quad \text{if} \quad p>1$$

and so the integral converges. On the other hand, if $p<1$, then $p-1<0$ and so

$$\frac{1}{t^{p-1}} = t^{1-p} \to \infty \quad \text{as} \quad t\to\infty$$

and the integral diverges. So,

$$\boxed{\int_1^\infty \frac{1}{x^p}dx \text{ is convergent if } p>1 \text{ and divergent if } p\leq 1.}$$

EXAMPLES: Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

1. $\displaystyle\int_{-\infty}^0 e^x dx$

2. $\displaystyle\int_0^\infty e^x dx$

3. $\displaystyle\int_{-\infty}^\infty x dx$

4. $\displaystyle\int_0^\infty (1-x)e^{-x}dx$

5. $\displaystyle\int_0^\infty \frac{dx}{x^2+4}$

6. $\displaystyle\int_{-\infty}^\infty \frac{dx}{e^x+e^{-x}}$

SOLUTIONS:

1. We have

$$\int_{-\infty}^{0} e^x dx = \lim_{t\to-\infty}\int_{t}^{0} e^x dx = \lim_{t\to-\infty} e^x\Big]_t^0 = \lim_{t\to-\infty}(e^0 - e^t) = (1-0) = 1 \quad \text{(convergent)}$$

2. We have

$$\int_{0}^{\infty} e^x dx = \lim_{t\to\infty}\int_{0}^{t} e^x dx = \lim_{t\to\infty} e^x\Big]_0^t = \lim_{t\to\infty}(e^t - e^0) = \infty \quad \text{(divergent)}$$

3. We have

$$\int_{-\infty}^{\infty} x dx = \int_{-\infty}^{1} x dx + \int_{1}^{\infty} x dx$$

The integral $\int_{1}^{\infty} x dx$ is divergent by the p-test, since $p = -1 \le 1$. Therefore $\int_{-\infty}^{\infty} x dx$ is divergent.

4. We first note that

$$\int (1-x)e^{-x}dx = \left[\begin{array}{c|c} 1-x=u & e^{-x}dx = dv \\ d(1-x) = du & -e^{-x} = v \\ -dx = du & \end{array}\right] = (1-x)(-e^{-x}) - \int (-e^{-x})(-dx)$$

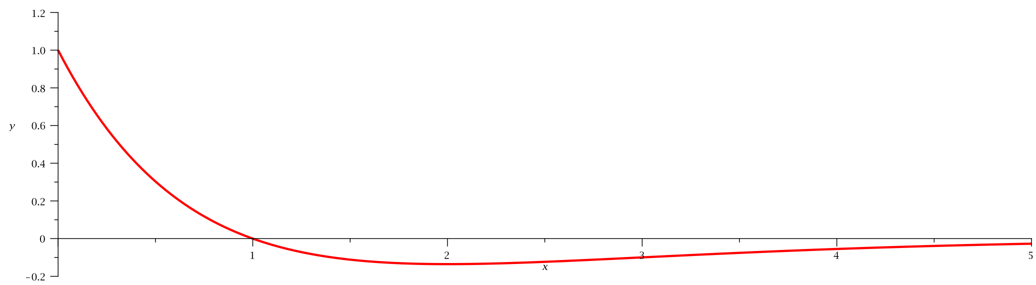
$$= (x-1)e^{-x} - \int e^{-x}dx = (x-1)e^{-x} + e^{-x} + C = xe^{-x} + C$$

We also note that

$$\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty}\frac{x}{e^x} = \lim_{x\to\infty}\frac{x'}{(e^x)'} = \lim_{x\to\infty}\frac{1}{e^x} = 0$$

by L'Hospital's Rule. Therefore

$$\int_{0}^{\infty}(1-x)e^{-x}dx = \lim_{t\to\infty}\int_{0}^{t}(1-x)e^{-x}dx = \lim_{t\to\infty} xe^{-x}\Big]_0^t = \lim_{t\to\infty}(te^{-t} - 0\cdot e^0) = 0 - 0 = 0 \quad \text{(convergent)}$$



5. We have

$$\int_0^\infty \frac{dx}{x^2+4} = \lim_{t\to\infty}\int_0^t \frac{dx}{x^2+4} = \lim_{t\to\infty}\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\Big]_0^t$$

$$= \lim_{t\to\infty}\frac{1}{2}\left(\tan^{-1}\left(\frac{t}{2}\right)-\tan^{-1}0\right) = \frac{1}{2}\left(\frac{\pi}{2}-0\right) = \frac{\pi}{4} \quad \text{(convergent)}$$

6. We first note that

$$\int_{-\infty}^\infty \frac{dx}{e^x+e^{-x}} = \int_{-\infty}^\infty \frac{e^x dx}{e^{2x}+1} = \int_{-\infty}^0 \frac{e^x dx}{e^{2x}+1} + \int_0^\infty \frac{e^x dx}{e^{2x}+1}$$

We also note that

$$\int \frac{e^x dx}{e^{2x}+1} = \left[\begin{array}{c} e^x = u \\ d(e^x) = du \\ e^x dx = du \end{array}\right] = \int \frac{1}{u^2+1}du = \tan^{-1}u + C = \tan^{-1}(e^x) + C$$

Therefore

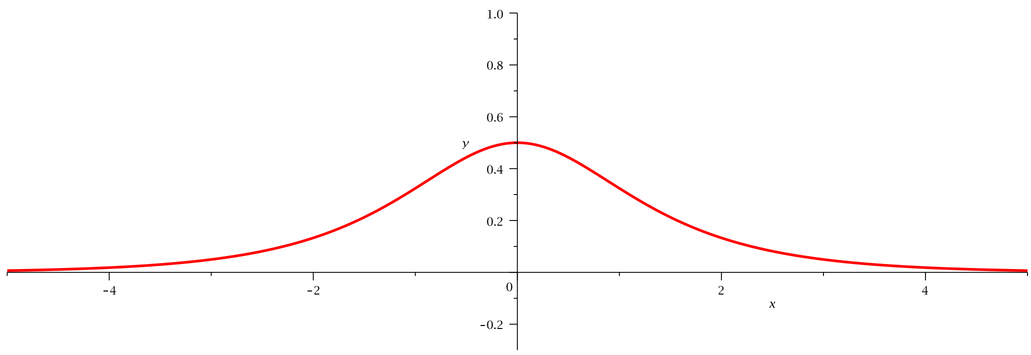
$$\int_{-\infty}^0 \frac{e^x dx}{e^{2x}+1} = \lim_{t\to-\infty}\int_t^0 \frac{e^x dx}{e^{2x}+1} = \lim_{t\to-\infty}\tan^{-1}(e^x)]_t^0 = \lim_{t\to-\infty}(\tan^{-1}(e^0)-\tan^{-1}(e^t)) = \frac{\pi}{4}-0 = \frac{\pi}{4}$$

and

$$\int_0^\infty \frac{e^x dx}{e^{2x}+1} = \lim_{t\to\infty}\int_0^t \frac{e^x dx}{e^{2x}+1} = \lim_{t\to\infty}\tan^{-1}(e^x)]_0^t = \lim_{t\to\infty}(\tan^{-1}(e^t)-\tan^{-1}(e^0)) = \frac{\pi}{2}-\frac{\pi}{4} = \frac{\pi}{4}$$

hence

$$\int_{-\infty}^\infty \frac{dx}{e^x+e^{-x}} = \frac{\pi}{4}+\frac{\pi}{4} = \frac{\pi}{2} \quad \text{(convergent)}$$



Type 2: Discontinuous Integrands

DEFINITION OF AN IMPROPER INTEGRAL OF TYPE 2:

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{t\to b^-}\int_a^t f(x)dx$$

if this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{t\to a^+}\int_t^b f(x)dx$$

if this limit exists (as a finite number).

The improper integral $\int_a^b f(x)dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$, then the improper integral $\int_a^b f(x)dx$ is defined as

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

It is said to converge if both terms converge and diverge if either term diverges.

EXAMPLES:

1. Evaluate $\int_1^2 \frac{dx}{1-x}$ if possible.

Solution: We first note that the given integral is improper because $f(x) = \frac{1}{1-x}$ has the vertical asymptote $x = 1$. We have

$$\int_1^2 \frac{dx}{1-x} = \lim_{t\to 1^+}\int_t^2 \frac{dx}{1-x} = \lim_{t\to 1^+} -\ln|1-x|\Big]_t^2 = \lim_{t\to 1^+}(-\ln 1 + \ln|1-t|) = -\infty \quad \text{(divergent)}$$

2. Evaluate $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$ if possible.

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Solution: We first note that the given integral is improper because $f(x) = \frac{1}{\sqrt{9-x^2}}$ has the vertical asymptotes $x = \pm 3$. We have

$$\begin{aligned}\int_0^3 \frac{dx}{\sqrt{9-x^2}} &= \lim_{t\to 3^-} \int_0^t \frac{dx}{\sqrt{9-x^2}} = \lim_{t\to 3^-} \sin^{-1}\left(\frac{x}{3}\right)\Big]_0^t \\ &= \lim_{t\to 3^-} \left(\sin^{-1}\left(\frac{t}{3}\right) - \sin^{-1} 0\right) = \sin^{-1} 1 - 0 = \frac{\pi}{2} \quad \text{(convergent)}\end{aligned}$$

3. Evaluate $\int_{-1}^1 \frac{dx}{x}$ if possible.

Solution: We first note that the given integral is improper because $f(x) = \frac{1}{x}$ has the vertical asymptote $x = 0$. We have

$$\int_{-1}^1 \frac{dx}{x} = \int_{-1}^0 \frac{dx}{x} + \int_0^1 \frac{dx}{x}$$

Since

$$\int_0^1 \frac{dx}{x} = \lim_{t\to 0^+} \int_t^1 \frac{dx}{x} = \lim_{t\to 0^+} \ln|x|\Big]_t^1 = \lim_{t\to 0^+} (\ln 1 - \ln|t|) = \infty$$

it follows that $\int_{-1}^1 \frac{dx}{x}$ is divergent.

A Comparison Test for Improper Integrals

COMPARISON TEST: Suppose that f and g are continuous functions with $f(x) \ge g(x) \ge 0$ for $x \ge a$.

(a) If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.

(b) If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.

EXAMPLE: The integral $\int_1^\infty \frac{dx}{e^x + x^2}$ is convergent, because $\frac{1}{x^2} > \frac{1}{e^x + x^2} > 0$ and $\int_1^\infty \frac{dx}{x^2}$ is convergent by the p-test, since $p = 2 > 1$.

EXAMPLE: Does the integral $\int\limits_1^\infty \frac{1}{xe^x}dx$ converge?

Solution: We have

$$0 < \frac{1}{xe^x} < \frac{1}{e^x}$$

Note that $\int\limits_1^\infty \frac{1}{e^x}dx$ is convergent, since

$$\int\limits_1^\infty \frac{1}{e^x}dx = \int\limits_1^\infty e^{-x}dx = \lim_{t\to\infty}\int\limits_1^t e^{-x}dx = \lim_{t\to\infty}[-e^{-x}]_1^t = \lim_{t\to\infty}(-e^{-t}+e^{-1}) = e^{-1}$$

Therefore the integral $\int\limits_1^\infty \frac{1}{xe^x}dx$ converges.

EXAMPLE: Does the integral $\int\limits_1^\infty \frac{dx}{\sqrt{x^3+1}}$ converge?

Solution: We have

$$0 < \frac{1}{\sqrt{x^3+1}} < \frac{1}{\sqrt{x^3}}$$

Note that $\int\limits_1^\infty \frac{1}{\sqrt{x^3}}dx$ is convergent by the p-test, since $p = 3/2 > 1$. Therefore the integral $\int\limits_1^\infty \frac{dx}{\sqrt{x^3+1}}$ converges.

EXAMPLE: Does the integral $\int\limits_3^\infty \frac{dx}{\sqrt[5]{x^2-x-3}}$ converge?

Solution: We have

$$0 < \frac{1}{\sqrt[5]{x^2}} < \frac{1}{\sqrt[5]{x^2-x-3}}$$

Note that $\int\limits_3^\infty \frac{1}{\sqrt[5]{x^2}}dx$ is divergent by the p-test, since $p = 2/5 \le 1$. Therefore the integral $\int\limits_3^\infty \frac{dx}{\sqrt[5]{x^2-x-3}}$ diverges.

EXAMPLE: Does the integral $\int\limits_2^\infty \frac{2+\sin x}{x-1}dx$ converge?

Solution: We have

$$0 < \frac{1}{x} < \frac{2+\sin x}{x-1}$$

Note that $\int\limits_2^\infty \frac{1}{x}dx$ is divergent by the p-test, since $p = 1 \le 1$. Therefore the integral $\int\limits_2^\infty \frac{2+\sin x}{x-1}dx$ diverges.