In most situations the quadratic equations such as: $x^2 + 8x + 5$, can be solved (factored) through the quadratic formula if factoring it out seems too hard. However, some of these problems may be solved faster by a method called: **Completing the square (or to complete the square)**.

First, we can use this technique for any equation that we can already solve by factoring.

For example we can complete the square for the equation $x^2 + 4x + 3$. This is a fairly easy equation to factor, but we will use the Complete the Square process to see how they relate. Completing the square also has the advantage of putting the equation in Standard Form: $a(x – h)^2 + k = c$, where (h, k) is the vertex point.

These are the steps to completing the square of a function:

Green numbers are the changed terms.

Step 1: Set the equation equal to zero if the function lacks an equal sign

$$x^2 + 4x + 3 = \mathbf{0}$$

Step 2: Subtract the constant term from both sides:

$$x^2 + 4x + 3 = 0$$
$$-3 \qquad -3$$
$$x^2 + 4x = \mathbf{-3}$$

Step 3: Divide all terms by leading coefficient.

When we complete the square we do not want to have any number other than one in front of our first term. That would complicate the process of completing the square.

In this case x^2 is not multiplied by any number other than one, so we can skip this step for this problem.

$$x^2 + \mathbf{4}x = -3$$

Step 4: Take the middle term, or "B" term from $Ax^2 + Bx + C$, and divide it by 2. (We always divide by 2 no matter what the B term is)

$$B = \mathbf{+4}$$
$$\frac{B}{2} = \frac{4}{2} = \mathbf{2}$$

Step 5: Square the $\frac{B}{2}$ number; this will give us our C term or constant value.

$$\left(\frac{B}{2}\right)^2 = \left(\frac{4}{2}\right)^2 = 2^2 = \mathbf{4}$$

Step 6: After we find out what this term should be, we add it to both sides of the equation.

This completes the square.

$$x^2 + 4x + 0 = -3$$
$$+4 \quad +4$$
$$\underbrace{x^2 + 4x + 4}_{(x+2)^2} = 1$$
$$(x + 2)^2 = 1$$

Step 7: Because the left side is a perfect square, we can take the square root both sides.

We are able to square root $x^2 + 4x + 4$ because $x^2 + 4x + 4 = (x + 2)^2$

The square root of the left side is $x+2$ while the square root of the right side is ± 1.

$$\sqrt{(x+2)^2} = \sqrt{1}$$
$$x + 2 = \pm 1$$

Step 8: Solve the equation for the two solutions.

A Quadratic has at most two possible solutions and this problem has two because of the $\pm$ sign. Since we have two solutions we will solve this problem as two equations.

This will give us the solutions -3 and -1. These are the same results we would get if we factored out the equation directly.

$$x + 2 = -1$$
$$-2 \quad -2$$
$$x = -3$$
$$x = -3$$

$$x + 2 = +1$$
$$-2 \quad -2$$
$$x = -1$$
$$x = -1$$

If we revisit steps 4 and 7, you may notice that we got +2 both when we divided B by 2 and when factored out the root of $x^2 + 4x + 4$ to $(x+2)^2$. We can always expect half our B term to be added to X… $(X \pm B/2)^2$

Completing the square is useful because it gives us an alternative to the quadratic formula and can even solve problems that the quadratic formula cannot.

While this previous problem solved may have been factored, here one example that needs to use this formula.

Example 1: Complete the square for the quadratic $x^2 - 6x + 7 = 0$

Our approach is nearly identical to our first problem; the main difference is that the arithmetic will be slightly more complex.

$$x^2 - 6x + 7 = 0$$

Step 1: Since this equation is equal to zero step 1 is satisfied.

$$x^2 - 6x + 7 = 0$$
$$\mathbf{-7} \quad \mathbf{-7}$$
$$x^2 - 6x \quad = \mathbf{-7}$$

Step 2: Subtract 7 (the constant) from both sides.

Step 3: Step 3 is satisfied, because we do not have a coefficient other than 1 in front of our leading variable.

$$\frac{B}{2} = \frac{-6}{2} = \mathbf{-3}$$

Step 4: Since B = –6, in Ax^2+Bx+c, divide it by 2

$$B = \mathbf{-6}$$

$$\left(\frac{B}{2}\right)^2 = \left(\frac{-6}{2}\right)^2 = (-3)^2 = \mathbf{9}$$

Step 5: Square $\frac{B}{2}$ to get our C term we need for a perfect square.

$$x^2 - 6x + 0 = -7$$
$$+9 \quad +9$$
$$\underbrace{x^2 - 6x + \mathbf{9}}_{(x-3)^2} = \mathbf{2}$$
$$(x - 3)^2 = 2$$

Step 6: Add $\left(\frac{B}{2}\right)^2$ to both sides.

$$\sqrt{(x-3)^2} = \sqrt{2}$$
$$\boldsymbol{x - 3 = \pm\sqrt{2}}$$
$$\boldsymbol{x = 3 \pm \sqrt{2}}$$

Step 7: Square root both sides.

We can square root $(x-3)^2$ into $(x-3)$.

We do not need to simplify $\sqrt{2}$ to a decimal, but we need to note that two answers exist: $+\sqrt{2}$ and $-\sqrt{2}$.

$$x = 3 \pm \sqrt{2}$$
$$x = 3 + \sqrt{2}$$
$$x = 3 - \sqrt{2}$$

Step 8: Break up x - 3 = $\pm\sqrt{2}$ into two equations and solve for x.

These last two problems were similar because we had x^2 as our leading term. No number was multiplied or divided to x^2 to changes its coefficient. This next problem will have this change.

Example 2: Factor $4y^2 + 4y = 9$ by completing the square.

Although the 4 out in front on the equation makes this problem slightly more complicated, we are prepared to solve this problem. We will follow the same steps as before with a few changes.

Step 1: Since this equation is set equal to a value, Step 1 is satisfied

$$4y^2 - 4y = 9$$

Step 2: Also, since the constant term is on the other side step 2 is also satisfied.

$$4y^2 - 4y = 9$$

Step 3: Divide all terms by leading coefficient. In this problem, we have a coefficient that is multiplied to our squared term. Because of this we will factor out +4 from each term. From here we can cancel out the +4 and further simplify.

$$4y^2 - 4y = 9$$
$$\frac{4y^2 - 4y}{4} = \frac{9}{4}$$
$$\frac{\cancel{4}\,1(y^2 - y)}{\cancel{4}\,1} = \frac{9}{4}$$
$$y^2 - y = \frac{9}{4}$$

Step 4: Divide B term by 2, Now that we factored out four from our equation

$$y^2 - y = \frac{9}{4}$$

$$B = -1$$

$$\frac{B}{2} = \frac{-1}{2} = -1/2$$

Step 5: And square $\frac{B}{2}$ to find the C term we need.

$$\left(\frac{B}{2}\right)^2 = \left(\frac{-1}{2}\right)^2 = \mathbf{1/4}$$

Step 6: Add 1/4 to both sides to make the left side of the equation a perfect square.

$$y^2 - y + 0 = \frac{9}{4}$$

$$+\frac{1}{4} \quad +\frac{1}{4}$$

$$y^2 - y + \frac{1}{4} = \frac{10}{4}$$

$$\left(y - \frac{1}{2}\right)^2 = \frac{10}{4}$$

Step 7: Square root both sides.

$$\sqrt{\left(y - \frac{1}{2}\right)^2} = \sqrt{\frac{10}{4}}$$

$$y - \frac{1}{2} = \pm\frac{\sqrt{10}}{2}$$

Step 8: Solve as two equations.

$$y = \frac{1}{2} + \frac{\sqrt{10}}{2}$$

$$y = \frac{1}{2} - \frac{\sqrt{10}}{2}$$