Lesson 03: Conditional Statements $P \rightarrow Q$

1. Truth table of the conditional statement

P	Q	$P \rightarrow Q$
T	T	T
T	**F**	**F**
F	T	T
F	F	T

P is called **antecedent**
Q is called **consequent**
Meaning of the conditional statement: The truth of P implies (leads to) the truth of Q

Note that when P is false the conditional statement is true no matter what the value of Q is. We say that in this case the conditional statement is **true by default or vacuously true.**

2. Representing the implication by means of disjunction

$$\mathbf{P \rightarrow Q \equiv \neg P \vee Q}$$

P	Q	$\neg$ P	**$P \rightarrow Q$**	**$\neg P \vee Q$**
T	T	F	T	T
T	F	F	**F**	**F**
F	T	T	T	T
F	F	T	T	T

Same truth tables

Usage:

1. To rewrite "OR" statements as conditional statements and vice versa (for better understanding)
2. To find the negation of a conditional statement using De Morgan's Laws

3. Rephrasing "or" sentences as "if-then" sentences and vice versa

Consider the sentence:
(1) "The book can be found in the library or in the bookstore".

Let

A = The book can be found in the library
B = The book can be found in the bookstore

Logical form of (1): **A V B**

Rewrite A V B **as a conditional statement**

In order to do this we need to use the commutative laws, the equivalence $\neg(\neg P) \equiv P$, and the equivalence $P \rightarrow Q \equiv \neg P \vee Q$

Thus we have:

$A \vee B \equiv \neg(\neg A) \vee B \equiv \neg A \rightarrow B$

The last expression $\neg A \rightarrow B$ is translated into English as
**"If the book cannot be found in the library,
it can be found in the bookstore**".

Here the statement "The book cannot be found in the library" is represented by $\neg A$

There is still one more conditional statement to consider.
$A \vee B \equiv B \vee A$ (commutative laws)

Then, following the same pattern we have:

$B \vee A \equiv \neg(\neg B) \vee A \equiv \neg B \rightarrow A$

The English sentence is: **"If the book cannot be found in the bookstore, it can be found in the library.**

We have shown that:

$A \vee B \equiv \neg(\neg A) \vee B \equiv \neg A \rightarrow B$
$A \vee B \equiv B \vee A \equiv \neg(\neg B) \vee A \equiv \neg B \rightarrow A$

Thus the sentence **"The book can be found in the library or in the bookstore"**
can be rephrased as:
"If the book cannot be found in the library, it can be found in the bookstore".
"If the book cannot be found in the bookstore, it can be found in the library.

4. **Negation of conditional statements**

Positive: The sun shines
Negative: The sun does not shine

Positive: " If the temperature is 250°F then the compound is boiling "
Negative: ?
In order to find the negation, we use De Morgan's Laws.

Let
P = the temperature is 250°F
Q = the compound is boiling

Positive: $P \rightarrow Q \equiv \neg P \vee Q$
Negative: $\neg (P \rightarrow Q) \equiv \neg (\neg P \vee Q) \equiv \neg (\neg P) \wedge \neg Q \equiv P \wedge \neg Q$

Negative: The temperature is 250°F however the compound is not boiling

IMPORTANT TO KNOW:

The negation of a disjunction is a conjunction.
The negation of a conjunction is a disjunction

The **negation of a conditional statement is a conjunction**, not another if-then statement

Question: Which logical connective when negated will result in a conditional statement?

5. **Necessary and sufficient conditions**

Definition:

"P is a **sufficient condition** for Q" means : **if P then Q, $P \rightarrow Q$**
"P is a **necessary condition** for Q" means: **if not P then not Q, $\sim P \rightarrow \sim Q$**
The statement **$\sim P \rightarrow \sim Q$ is equivalent to $Q \rightarrow P$**

Hence given the statement **$P \rightarrow Q$,**
P is a sufficient condition for Q, and Q is a necessary condition for P.

Examples:

If *n* is divisible by 6 then *n* is divisible by 2.
The sufficient condition to be divisible by 2 is to be divisible by 6.
The necessary condition to be divisible by 6 is to be divisible by 2

If n is odd then n is an integer.
The sufficient condition to be an integer to be odd.
The necessary condition to be odd is to be an integer.

If and only if - the biconditional

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

$P \leftrightarrow Q$ is true whenever P and Q have same values. Otherwise it is false.

This means that **both $P \rightarrow Q$ and $Q \rightarrow P$ have to be true**

P	Q	**P → Q**	**Q → P**	P↔Q
T	**T**	**T**	**T**	**T**
T	F	F	T	F
F	T	T	F	F
F	**F**	**T**	**T**	**T**

P if and only if Q means that P is a necessary and sufficient condition for Q, and Q is a necessary and sufficient condition for P

Example:

(1) If n is divisible by 6 then n is divisible by 2 and n is divisible by 3.
(2) If n is divisible by 2 and n is divisible by 3 then n is divisible by 6.

(1) and (2) can be combined :

(3) n is divisible by 6 if and only if (iff) n is divisible by 2 and n is divisible by 3.

Atomic statements in (3) are:
P = n is divisible by 6
Q = n is divisible by 2
R = n is divisible by 3

Representation of (3) in logical form: $P \leftrightarrow Q \land R$

6. Contrapositive, converse and inverse

Consider the conditional statement:
(1) "If I need detergents, I go to WalMart"

What about
(2) "If I go to WalMart, I need detergents"
(3) "If I don't need detergents I don't go to WalMart."
(4) "If I don't go to WalMart I don't need detergents."

Are they equivalent to (1)?

We shall show now that (1) is equivalent to (4), and (2) is equivalent to (3), however (1) is not equivalent to (2) or to (3).

Contrapositive

Definition: The expression $\sim Q \rightarrow \sim P$ is called **contrapositive** of $P \rightarrow Q$

The conditional statement $P \rightarrow Q$ and its contrapositive $\sim Q \rightarrow \sim P$ **are equivalent**.
The proof is done by comparing the truth tables

The truth table for $P \rightarrow Q$ and $\neg Q \rightarrow \neg P$ is:

P	Q	¬ P	¬ Q	P→ Q	¬ Q → ¬ P
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

We can also prove the equivalence by using the disjunctive representation:

$$P \rightarrow Q \equiv \neg P \vee Q \equiv Q \vee \neg P \equiv \neg (\neg Q) \vee \neg P \equiv \neg Q \rightarrow \neg P$$

Thus, the contrapositive of
"If I need detergents, I go to WalMart"
is
"If I don't go to WalMart I don't need detergents."
and hence they are equivalent.

What is the meaning of equivalence and how do we use it?
Consider the statement "If the lights are red, cars stop"

Let P = Lights are red, Q = Cars stop.

The logical form is: $P \rightarrow Q$. Its equivalent contrapositive is $\neg Q \rightarrow \neg P$, which means:

"If cars don't stop, the lights are not red." Then, when we approach an intersection, we can figure out what the lights are even if we don't see them, just by watching what cars do - if they move, the lights are not red.

However, if we see that cars stop, would that mean that the lights are red?
The answer is no, as there may be a pedestrian, or there may be an accident.
If we think that the answer is "yes", then we wrongly assume that the conditional statement $Q \rightarrow P$ (i.e. "if the cars stop then the lights are red") is equivalent to $P \rightarrow Q$.

$Q \rightarrow P$ is called **converse** of $P \rightarrow Q$, discussed in the next section, and they are not equivalent.

Converse and inverse

Definition: The converse of $P \to Q$ is the expression $Q \to P$

Definition: The inverse of $P \to Q$ is the expression $\sim P \to \sim Q$

Neither the converse nor the inverse are equivalent to the original implication.
Compare the truth tables and you will see the difference**.**

P	Q	¬ P	¬ Q	P→ Q	Q→ P	**¬ P → ¬ Q**
T	T	F	F	T	T	T
T	F	F	T	**F**	**T**	**T**
F	T	T	F	**T**	**F**	**F**
F	F	T	T	T	T	T

However, the converse is equivalent to the inverse - the columns for Q→ P and
¬ P → ¬ Q are the same.

If you look closer, you will find out that the converse expression is contrapositive to the inverse expression and vice versa.

Examples

(1) "If I need detergents, I go to WalMart"

converse: "If I go to WalMart, I need detergents"
inverse: "If I don't need detergents I don't go to WalMart."

Some notes about the meaning of the conditional statements:

A conditional statement consists of two other statements, say P and Q.
Formally, P and Q may have no semantic relation, as in
"If we have a party, then this book is expensive"

In real world however conditional statements are used generally to express a cause-effect relation or a property.

a. cause-effect relation

If P then Q means that P is the cause of Q

If you have an insurance you will feel much safer.
If you jump from the fifth floor you will break your neck.
If the program contains an error it will not run properly.

Sometimes **If P then Q** means that P is the effect of Q .

If you have high temperature you are sick.

b. properties of objects that belong to some hierarchy
If Koko is a bird, Koko can fly
c. Sometimes when used informally, the conditional statements are actually biconditional statements, i.e. their meaning is 'If P then Q" and "If Q then P"
An example is "If a number is positive, then it is greater than zero".

7. P only if Q and P unless Q

A. Consider the statement:

n is divisible by 6 **only if** n is divisible by 2.

Let

P = n is divisible by 6
Q = n is divisible by 2

Which is the correct logical representation?:

$P \rightarrow Q$ or $Q \rightarrow P$

i.e. If n is divisible by 6 then n is divisible by 2.
Or If n is divisible by 2 then n is divisible by 6

The answer is: If n is divisible by 6 then n is divisible by 2.

P Only if Q is equivalent to **If P then Q,** and also to **If not Q then not P**
(note that the two are contrapositives)

The meaning is: P will occur only if Q occurs, i.e. if Q does not occur then P will not occur. The latter means: **If not Q then not P,** which is equivalent to **If P then Q**

Example:

Tom will buy a house only if he gets a promotion. P only if Q

Otherwise Tom will not buy a house, i.e.

If Tom does not get a promotion he will not buy a house: $\sim Q \rightarrow \sim P$

This however is the contrapositive of $P \rightarrow Q$

i.e. **If Tom buys a house then he has been promoted**.

Note the ordering of the events in time. Time is not present in the logical expressions so formally it is correct to say:

If Tom buys a house then he will get a promotion.

However when translating logical expressions we always consider the plausible time sequencing of the events.

In language, if the event P precedes Q, we use "if P then Q".
If Q precedes P we say "P only if Q".

B. Consider the statement

I will have financial problems **unless** I find a job

Let P = I will have financial problems
Q = I find a job

Which of the following implications conveys the meaning of the above statement:

(1) $P \rightarrow Q$	(5) $Q \rightarrow P$
(2) $\sim P \rightarrow Q$	(6) $\sim Q \rightarrow P$
(3) $P \rightarrow \sim Q$	(7) $Q \rightarrow \sim P$
(4) $\sim P \rightarrow \sim Q$	(8) $\sim Q \rightarrow \sim P$

P unless Q means **if not Q then P, $\sim Q \rightarrow P$**

If I don't find a job, I will have financial problems.

Thus given a statement $P \rightarrow Q$ there are several ways to rephrase the corresponding English sentence.

Example: If *n* is divisible by 6 then *n* is divisible by 2.

P = *n* is divisible by 6
Q = *n* is divisible by 2.

The statement is: $P \rightarrow Q$

Disjunctive form: $\sim P \vee Q$
Either *n* is not divisible by 6 or *n* is divisible by 2.

Contrapositive: $\sim Q \rightarrow \sim P$
If *n* is not divisible by 2 then *n* is not divisible by 6

Only if: P only if Q
n is divisible by 6 only if *n* is divisible by 2

$\sim Q$ only if $\sim P$ (derived from $\sim Q \rightarrow \sim P$)
n is not divisible by 2 only if *n* is not divisible by 6

Unless: $\sim P$ unless Q

n is not divisible by 6 unless *n* is divisible by 2

This is obtained in the following way:
A unless B is equivalent to $\sim B \rightarrow A$.
On the other hand, $P \rightarrow Q = \sim Q \rightarrow \sim P$.
Substituting B with Q and A with $\sim P$, we obtain $\sim P$ unless Q.

SUMMARY

1. The implication $P \rightarrow Q$ is false only when P is T and Q is F. In all other cases it is true

2. P is a sufficient condition for Q
   Q is a necessary condition for P

3. Representing the implication as a disjunction:

   $$P \rightarrow Q \equiv \neg P \lor Q$$

4. Negation of the conditional statement: use De Morgan's laws and negate its equivalent disjunction:

   $$\neg (P \rightarrow Q) \equiv \neg(\neg P \lor Q) \equiv \neg(\neg P) \land \neg Q \equiv P \land \neg Q$$

This is the detailed writing of the transformations. If the implication consists of simple statements, we can write

$$\neg (P \rightarrow Q) \equiv P \land \neg Q$$

This means: the "condition" part P stays unchanged, while the "conclusion" part Q is negated.

5. P only if Q is: $P \rightarrow Q$
   P unless Q is $\neg Q \rightarrow P$

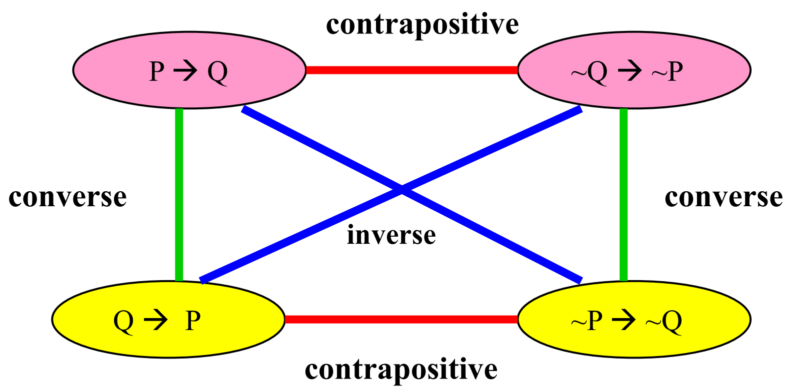
6. $P \rightarrow Q$ is equivalent to its **contrapositive** $\neg Q \rightarrow \neg P$

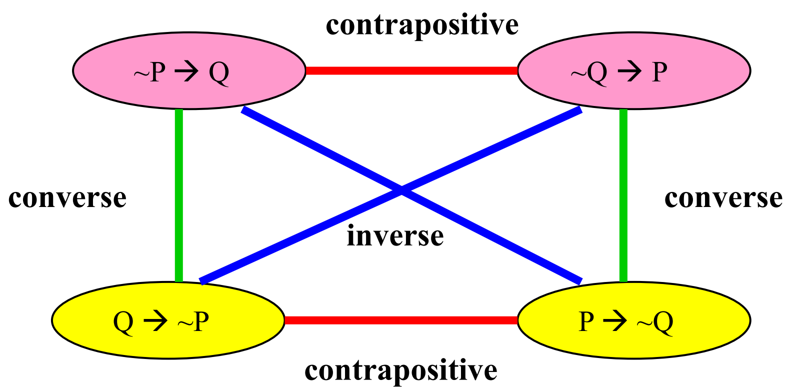
   Converse of $P \rightarrow Q$ is $Q \rightarrow P$

   Inverse of $P \rightarrow Q$ is $\neg P \rightarrow \neg Q$

   The converse and inverse are mutually equivalent

The following diagram pictures the eight possible combinations of P, Q, ~P and ~Q in an implication.





Ovals with same color are equivalent.

Exercise:

If today is Easter, tomorrow is Monday

1. Identify atomic propositions:

   P:
   Q:

2. Write the logical form

3. Rephrase the sentence as an "OR" sentence (disjunction)

4. Rephrase the sentence as another "If Then" sentence (implication)

5. Write the negation of the sentence as a logical form and translate the negation in English

6. Fill in:

   ……………………………. is a sufficient condition for ……………………………

   ……………………………. is a necessary condition for ……………………………

7. Rephrase using “only if”:

8. Rephrase using “unless”:

9. Write the contrapositive and translate it in English

10. Write the converse and translate it in English

11. Write the inverse and translate it in English.