Ordinary Annuities

Chapter 10
Learning Objectives

After completing this chapter, you will be able to:

> Define and distinguish between ordinary simple annuities and ordinary general annuities.
> Calculate the future value and present value of ordinary simple annuities.
> Calculate the fair market value of a cash flow stream that includes an annuity.
> Calculate the principal balance owed on a loan immediately after any payment.
> Calculate the present value and period of deferral of a deferred annuity.
> Calculate the interest rate per payment interval in a general annuity.
Annuity - A series of equal payments at regular intervals

Term of the annuity - the time from the beginning of the first payment period to the end of the last payment period.

Future value of annuity - the future dollar amount of a series of payments plus interest

Present value of an annuity - the amount of money needed to invest today in order to receive a stream of payments for a given number of years in the future
**Terminology**

- \( PMT \) = Amount of each payment in an annuity

- \( n \) = Number of payments in the annuity

- **payment interval** is the time between successive payments in an annuity.

- **ordinary annuities** are ones in which payments are made at the *end* of each payment interval.
Suppose you obtain a personal loan to be repaid by 48 equal monthly payments.

- The payment interval is 1 month.
- The term of the annuity is 48 months or 4 years.
- The first payment will be due 1 month after you receive the loan—i.e., at the end of the first payment interval.
- The payments form an ordinary annuity.
**Figure 10.1**

Time Diagram for an n-Payment Ordinary Annuity

Payment interval

0  1  2  3  \ldots  n-1  n

PMT  PMT  PMT  PMT  PMT

Interval number

Term of the annuity
Ordinary Simple Annuities:
The payment interval equals the compounding interval.

Eg. Monthly payments and interest is compounded monthly.

Ordinary General Annuities:
The payment interval differs from the compounding interval.

Eg. Monthly payments, but interest is compounded semi-annually.
Future Value of an Ordinary Simple Annuity

the sum of the future values of all the payments

\[ FV = PMT \left[ \frac{(1 + i)^n - 1}{i} \right] \]
Future Value of an Ordinary Simple Annuity

the sum of the future values of all the payments

Interval number

$1000 (1.04)

$n = 1$

$1000 (1.04)^2$

$n = 2$

$1000 (1.04)^3$

$n = 3$

Sum = $FV$ of annuity

Just like compound interest!
The Future Value of an Ordinary Simple Annuity

Assume 4 $1000 payments with interest at 4%

Sum = $FV of annuity
Figure 10.2
The Future Value of a Four-Payment Ordinary Simple Annuity

Assume 4 $1000 payments with interest at 4%

Sum = $V of annuity

\[ FV \text{ of annuity} = \$1000 + \$1000(1.04) + \$1000(1.04)^2 + \$1000(1.04)^3 \]
\[ = \$1000 + \$1040 + \$1081.60 + \$1124.86 \]
\[ = \$4246.46 \]
Suppose that you vow to save $500/month for the next four months, with your first deposit one month from today. If your savings can earn 3% pa converted monthly, determine the total in your account four months from now.

Now

<table>
<thead>
<tr>
<th>Month</th>
<th>Now</th>
<th>$500</th>
<th>$500</th>
<th>$500</th>
<th>$500</th>
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<td>0</td>
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<td></td>
</tr>
<tr>
<td>1</td>
<td>$500</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>$500</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td>$500</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td>$500</td>
<td></td>
</tr>
</tbody>
</table>

$500 (1 + .03/12)

$500 (1 + .03/12)^2

$500 (1 + .03/12)^3

Sum = $2,007.51

Result

$500.00
$501.25
$502.50
$503.76
$2,007.51
Now imagine that you save $500 every month for the next three years. Although the same logic applies, I certainly don’t want to do it this way!

Let’s start with the financial calculator method.

Since your account was empty when you began, PV = 0

n = 3 yrs * 12 pay’ts per year

= 36

FV = $18810.28

Financial Calculator sol’n:

3/12= I/Y

36 n

0 PV

500 +/- PMT

comp FV
Cash Flow Sign Convention

Keep in mind that when you are making payments, or even making deposits to savings, these are cash outflows, and therefore the values must be negative.
Suppose that you vow to save $500/month for the next four months, with your first deposit one month from today. If your savings can earn 3% pa converted monthly, determine the total in your account four months from now.

Since your account was empty when you began, PV = 0

\[ n = 4 \text{ payments} \]

\[ \text{FV} = \$2007.51 \]
Or, solving using the formula,

\[ FV = PMT \times \left( \frac{(1 + i)^n - 1}{i} \right) \]

PMT = $500
n = 4
i = 0.03/12

\[ \frac{0.03/12}{1} \times 4 = \]

\[ \frac{-1}{rcl} \times 500 = \]

$2007.51
Or, when investing for 3 full years:

\[ FV = PMT \left[ \frac{(1 + i)^n - 1}{i} \right] \]

**Calculation:**

- \( PMT = $500 \)
- \( n = 3 \times 12 = 36 \)
- \( i = 0.03/12 \)

\[ \frac{0.03/12}{1} = \text{sto} \]

\[ 1 + 36 \]

\[ 1 = y \]

\[ / \text{rcl} \times 500 = \]

**Result:** $18810.28
Figure 10.3
Contribution of Each Payment to an Annuity’s Future Value

Future Values

$14.64

$13.31

$12.10

$11.00

$10.00

$61.05

1st Payment
2nd Payment
3rd Payment
4th Payment
5th Payment

MH Ryerson
Suppose you decide to save $75/month for the next three years. If you invest all of these savings in an account which will pay you 7% pa compounded monthly, determine:

a) the total in the account after 3 years
b) the amount you deposited
c) the amount of interest earned

**Financial Calculator sol’n:**

- **PMT = - 75**
- **I/Y = 7/12**
- **n = 3*12 = 36**
- **PV = 0**
- **FV = ?**

**FV = $2994.76**

**Total deposits**

\[ = 75 \times 36 \]

\[ = $2700.00 \]

**Interest earned**

\[ = 2994.76 - 2700 \]

\[ = $294.76 \]
Present Value of an Ordinary Simple Annuity

the sum of the present values of all the payments

\[ PV = \frac{PMT[1 - (1 + i)^{-n}]}{i} \]
Figure 10.4
The Present Value of a Four-Payment Ordinary Simple Annuity

$1000 (1.04)^{-1}
$1000 (1.04)^{-2}
$1000 (1.04)^{-3}
$1000 (1.04)^{-4}

Sum = $PV$ of annuity
Figure 10.4
The Present Value of an Ordinary Simple Annuity

**Sum = PV of annuity**

\[
PV = 1000 (1.04)^{-1} + 1000 (1.04)^{-2} + 1000 (1.04)^{-3} + 1000 (1.04)^{-4}
\]

\[
= 961.54 + 924.56 + 889.00 + 854.80
\]

\[
= 3629.90
\]
You overhear your buddy saying he is repaying a loan at $500 every month for the next four months. The interest rate he has been charged is 12% pa compounded monthly. Figure out the size of the loan, and the amount of interest involved.

Interest charged = 2000 - 1950.98 = $49.02

Financial Calculator sol’n:

\[
\begin{align*}
12/12 & = I/Y \\
4 & = n \\
0 & = FV \\
500 & = PMT \\
\text{comp} & = PV
\end{align*}
\]
You overhear your buddy saying he is repaying a loan at $500 every month for the next four months. The interest rate he has been charged is 12% pa compounded monthly. Figure out the size of the loan, and the amount of interest involved.

Solving the same question using the formula:

\[
PV = PMT \left[ \frac{1 - (1 + i)^{-n}}{i} \right]
\]

\[
.12 / 12 = sto + 1 = y^x 4 +/- = -1 = +/- / rcl * 500 =
\]

\[
PV = PMT[1 - (1+ i)^{-n}] \quad i
\]

Interest charged =

2000 - 1950.98 = $49.02
Figure 10.5
Contribution of Each Payment to an Annuity’s Present Value

Present values

$9.09
$6.83
$7.51
$8.26
$37.91

1st payment
2nd payment
3rd payment
4th payment
5th payment

$10.00

$6.21
5 years

$6.83
4 years

$7.51
3 years

$8.26
2 years

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Deferred Annuities

A **deferred annuity** may be viewed as an **ordinary** annuity that does not begin until a time interval (named the **period of deferral**) has passed.

\[ d = \text{Number of payment intervals in the period of deferral} \]

**Two-step procedure to find PV:**

> Calculate the present value, \( PV_1 \), of the payments at the end of the period of deferral—this is just the PV of an ordinary annuity.

> Calculate the present value, \( PV_2 \), of the step 1 amount at the beginning of the period of deferral.
If this same buddy doesn’t begin to repay his loan for another 11 months, at a rate $500 every month for four months. The interest rate is still 12%pa compounded monthly. Determine the size of the loan.

The deferral period is 10 months

PV = $1950.98

This is the value 10 months from now

Today’s value:

1950.98 +/- FV

150.98 

$1766.20

Financial Calculator sol’n:

12/12= I/Y

4 n

0 FV

500 +/- PMT

10 n

comp PV

comp PV
General Annuities

\[ c = \frac{\text{number of compoundings per year}}{\text{number of payments per year}} \]

Use \[ i_2 = (1+i)^c - 1 \] to calculate the equivalent periodic rate that matches the payment interval.

Use this equivalent periodic rate as the value for “\( i \)” in the appropriate simple annuity formula, or as the value entered into the \( i \) memory of the financial calculator.
Suppose you decide to save $50/month for the next three years. If you invest all of these savings in an account which will pay you 7% pa compounded semi-annually, determine the total in the account after 3 years.
Suppose you decide to save $50/month for the next three years. If you invest all of these savings in an account which will pay you 7% pa compounded semi-annually, determine the total in the account after 3 years.

Note the differing compounding frequency and payment intervals.

This is a GENERAL annuity and so we need to calculate $c$

$$c = \frac{\text{number of compoundings per year}}{\text{number of payments per year}} = \frac{2}{12} = .16666$$

Store it!
Suppose you decide to save $50/month for the next three years. If you invest all of these savings in an account which will pay you 7% pa compounded semi-annually, determine the total in the account after 3 years.

\[
PMT = -50 \\
PV = 0 \\
FV = ? \\
n = 3 \times 12 = 36 \\
i = \frac{0.07}{2} \\
c = \frac{2}{12} = 0.16666 \\
i_2 =
\]

Once we have \( c \), we need to determine \( i_2 \).

\[
i_2 = (1 + i)^c - 1
\]

\[
\frac{2}{12} = \text{sto} \\
\frac{0.07}{2} = + 1 = y^x \text{ rcl} = \\
- 1 = \times 100 = \\
\]

\[
FV = $1993.51
\]
Tip: Improving the Accuracy of Calculated Results

- the value for \( c \) can be a repeating decimal
- when this happens, save \( c \) in memory
- your calculator then retains at least two more digits than you see in the display.
- when you need the exponent for the \( y^x \) function, recall the value for \( c \) from the memory
- the value for \( i_2 \) should be saved in memory as soon you calculate it. Whenever \( i_2 \) is needed in a subsequent calculation, recall it from the memory.
Reid David made annual deposits of $1,000 to Fleet Bank, which pays 6% interest compounded annually. After 4 years, Reid makes no more deposits. What will be the balance in the account 10 years after the last deposit?

Future value of an annuity

\[ FV = 4374.62 \]

Future value of a lump sum

\[ FV = 7834.03 \]
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