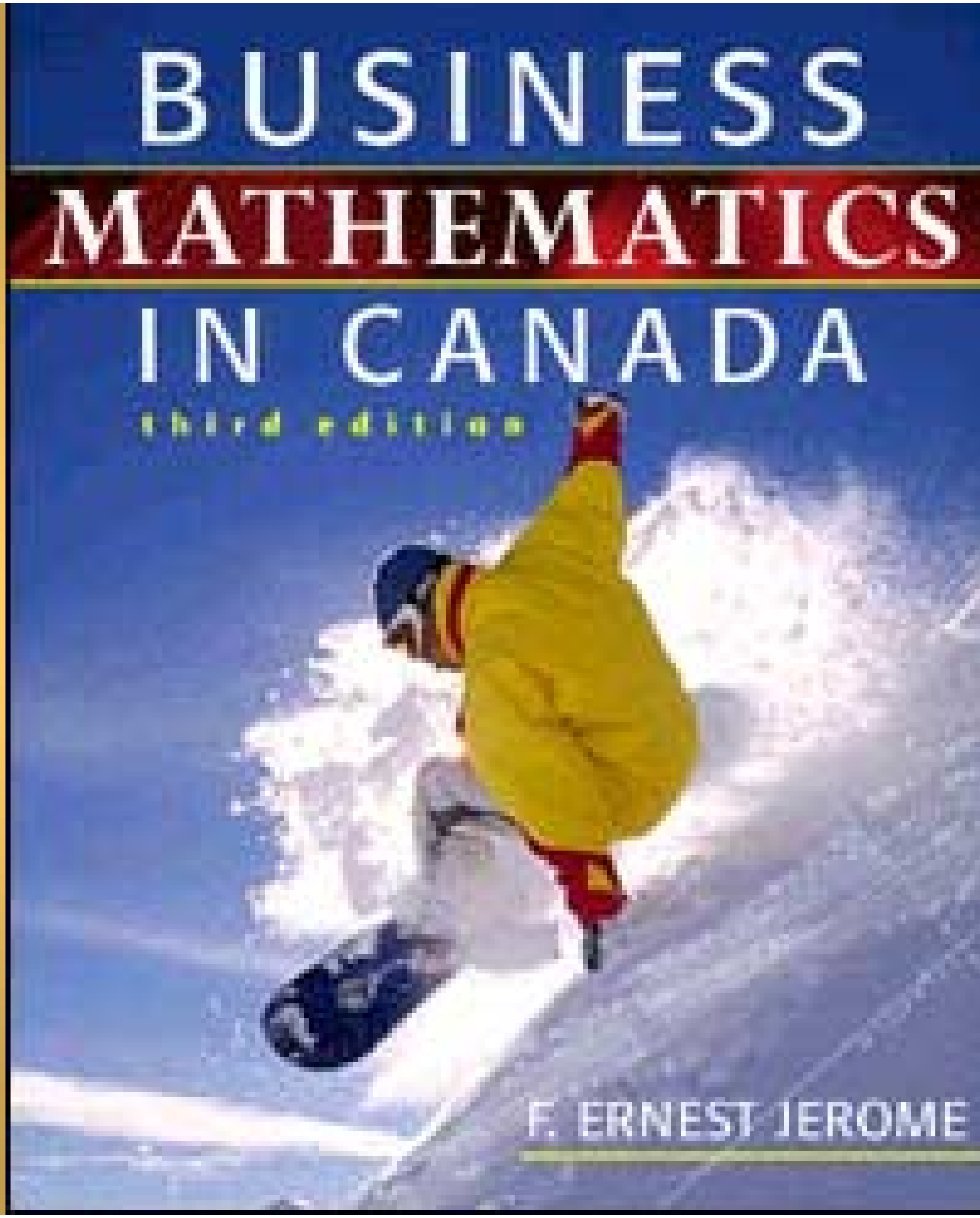

BUSINESS
MATHEMATICS
IN CANADA
third edition
F. ERNEST JEROME

Ordinary Annuities

Chapter 10

Learning Objectives

After completing this chapter, you will be able to:

- Define and distinguish between ordinary simple annuities and ordinary general annuities.
- Calculate the future value and present value of ordinary simple annuities.
- Calculate the fair market value of a cash flow stream that includes an annuity.
- Calculate the principal balance owed on a loan immediately after any payment.
- Calculate the present value and period of deferral of a deferred annuity
- Calculate the interest rate per payment interval in a general annuity.

Annuity - A series of equal payments at regular intervals

Term of the annuity - the time from the beginning of the first payment period to the end of the last payment period.

Future value of annuity - the future dollar amount of a series of payments plus interest

Present value of an annuity - the amount of money needed to invest today in order to receive a stream of payments for a given number of years in the future

Terminology

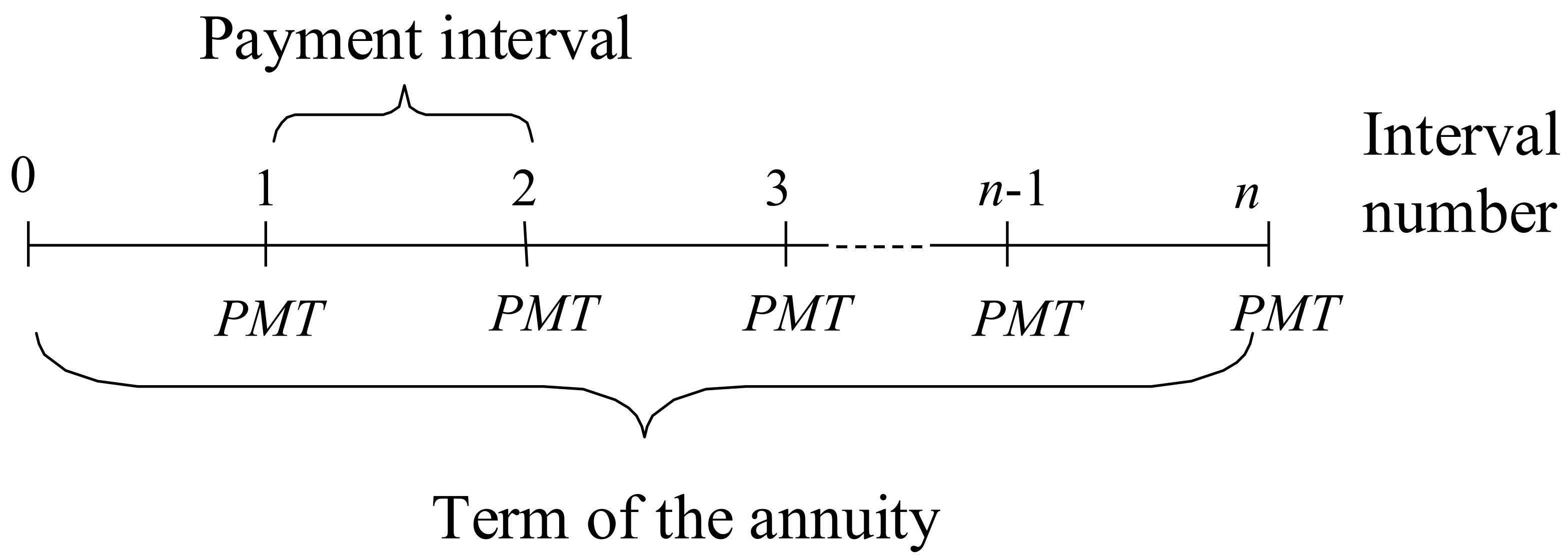
- *PMT* = Amount of each payment in an annuity
  - *n* = Number of payments in the annuity
- **payment interval** is the time between successive payments in an annuity.
- **ordinary annuities** are ones in which payments are made at the *end* of each payment interval.

Terminology

Suppose you obtain a personal loan to be repaid by 48 equal monthly payments.

- The *payment interval* is 1 month
- the *term of the annuity* is 48 months or 4years.
- The first payment will be due 1 month after you receive the loan—i.e.., at the *end* of the first payment interval.
- the payments form an *ordinary annuity*.

Figure 10.1
Time Diagram for an n-Payment Ordinary Annuity



Ordinary Annuities

Ordinary Simple Annuities:
The payment interval equals the compounding interval.

Eg. Monthly payments and interest is compounded monthly

Ordinary General Annuities:
The payment interval differs from the compounding interval.

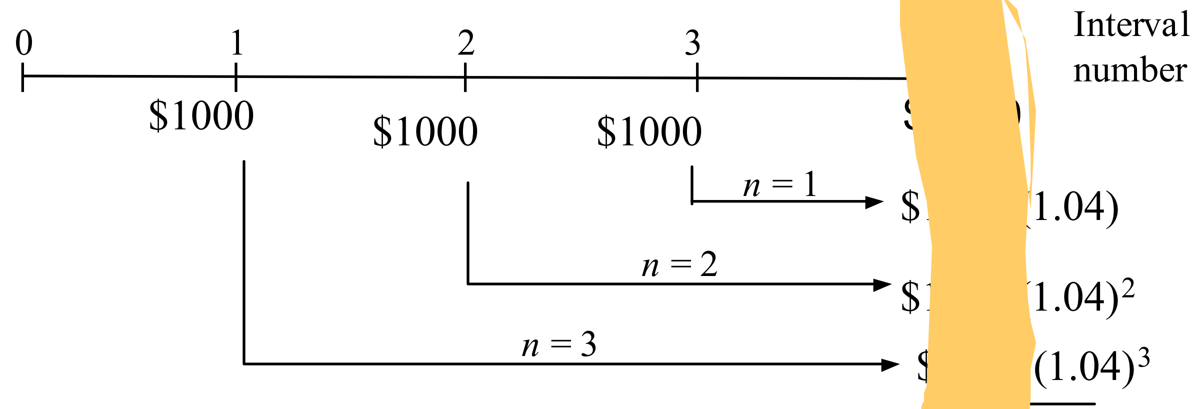
Eg. Monthly payments, but interest is compounded semi-annually.

Future Value of an Ordinary Simple Annuity

the sum of the future values of all the payments

$$FV = PMT \left[\frac{(1+i)^n - 1}{i} \right]$$

Future Value of an Ordinary Simple Annuity



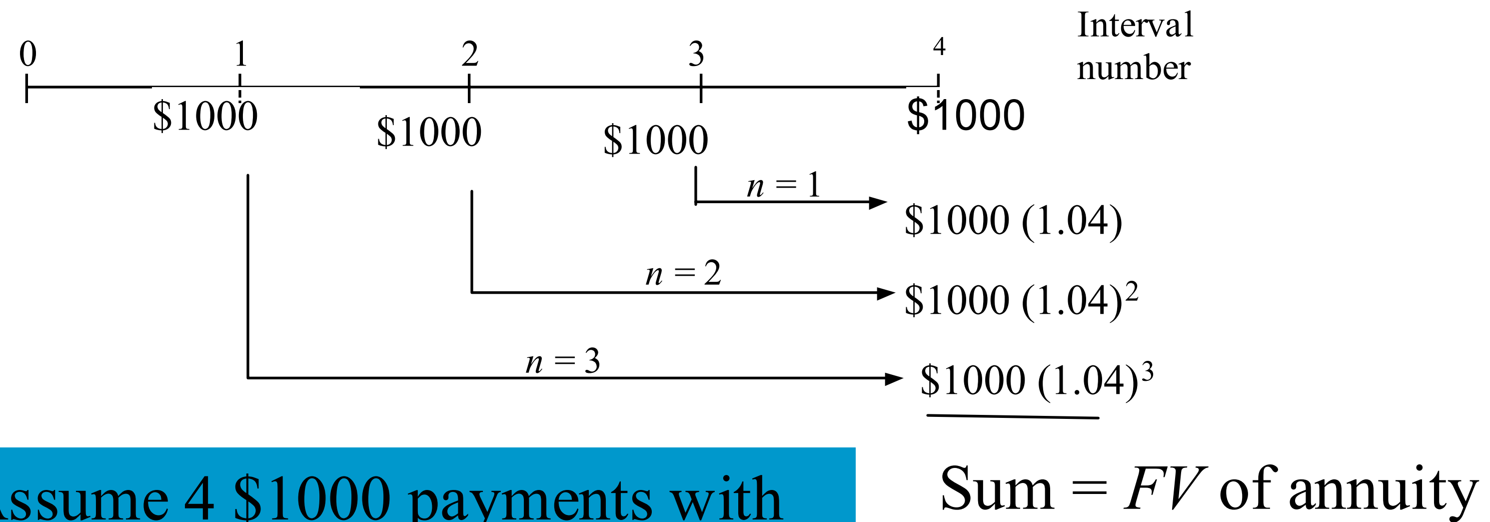
Sum = *FV* of annuity

Just like compound interest!

the sum of the future values of all the payments

Figure 10.2

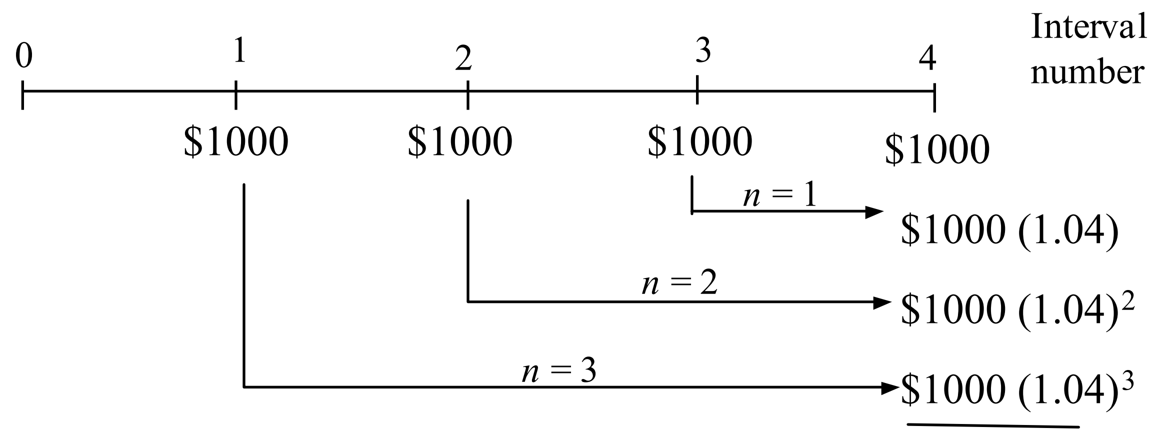
The Future Value of an Ordinary Simple Annuity



Assume 4 $1000 payments with interest at 4%

Figure 10.2

The Future Value of a Four-Payment Ordinary Simple Annuity

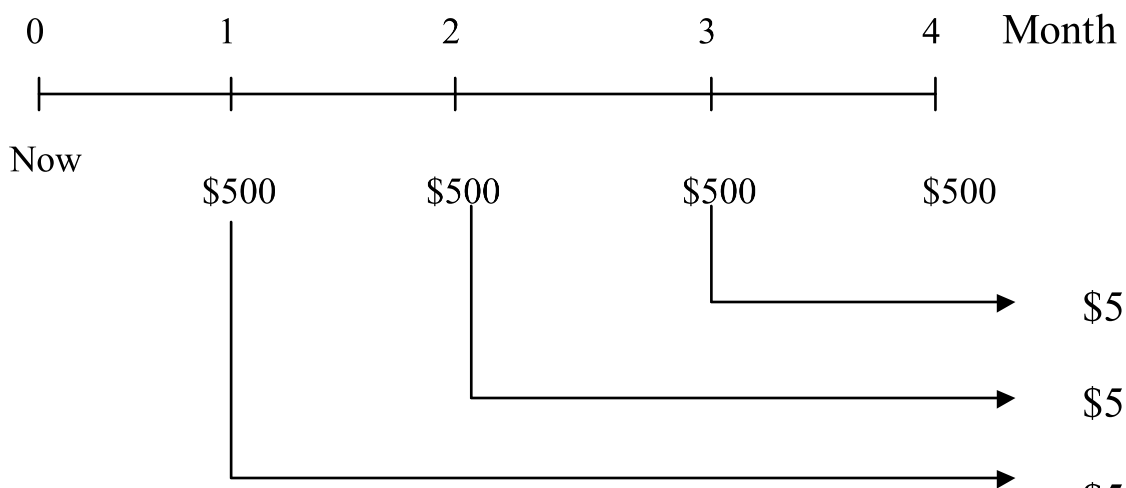


Sum = *FV* of annuity

Assume 4 $1000 payments with interest at 4%

$$
\begin{aligned}
FV \text{ of annuity} &= \$1000 + \$1000(1.04) + \$1000(1.04)^2 + \$1000(1.04)^3 \\
&= \$1000 + \$1040 + \$1081.60 + \$1124.86 \\
&= \$4246.46
\end{aligned}
$$

Suppose that you vow to save $500/month for the next four months, with your first deposit one month from today. If your savings can earn 3% pa converted monthly, determine the total in your account four months from now.



	Result $
	500.00
$\$500\,(1 + .03/12)$	**501.25**
$\$500\,(1 + .03/12)^2$	**502.50**
$\$500\,(1 + .03/12)^3$	**503.76**
Sum = *FV* of annuity	**$2,007.51**

Now imagine that you save \$500 every month for the next three years. Although the same logic applies, I certainly don't want to do it this way!

Let's start with the financial calculator method

Since your account was empty when you began, PV = 0

n = 3 yrs * 12 pay'ts per year

= 36

Financial Calculator sol'n:

3/12=	I/Y
36	n
0	PV
500 +/-	PMT
comp	FV

FV = $18810.28

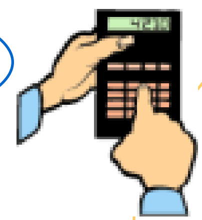
Cash Flow Sign Convention

Keep in mind that when you are making payments, or even making deposits to savings, these are cash outflows, and therefore the values must be negative.

Suppose that you vow to save $500/month for the next four months, with your first deposit one month from today. If your savings can earn 3% pa converted monthly, determine the total in your account four months from now.

Applying our new calculator solution……

Financial Calculator sol'n:



3/12 = I/Y

4 n

0 PV

500 +/- PMT

comp FV

Since your account was empty when you began, PV = 0

n = 4 payments

FV = **$2007.51**

Or, solving using the formula,

$$FV = PMT\frac{[(1+ i)^n - 1]}{i}$$

PMT = $500

n = 4

i = .03/12

.03/12 = sto

+ 1 = y^x 4 =

- 1 =

/ rcl * 500 =

$2007.51

Or, when investing for 3 full years:

$$FV = PMT\left[\frac{(1+i)^n - 1}{i}\right]$$

PMT = $500

n = 3 * 12 = 36

i = .03/12

.03/12 = sto

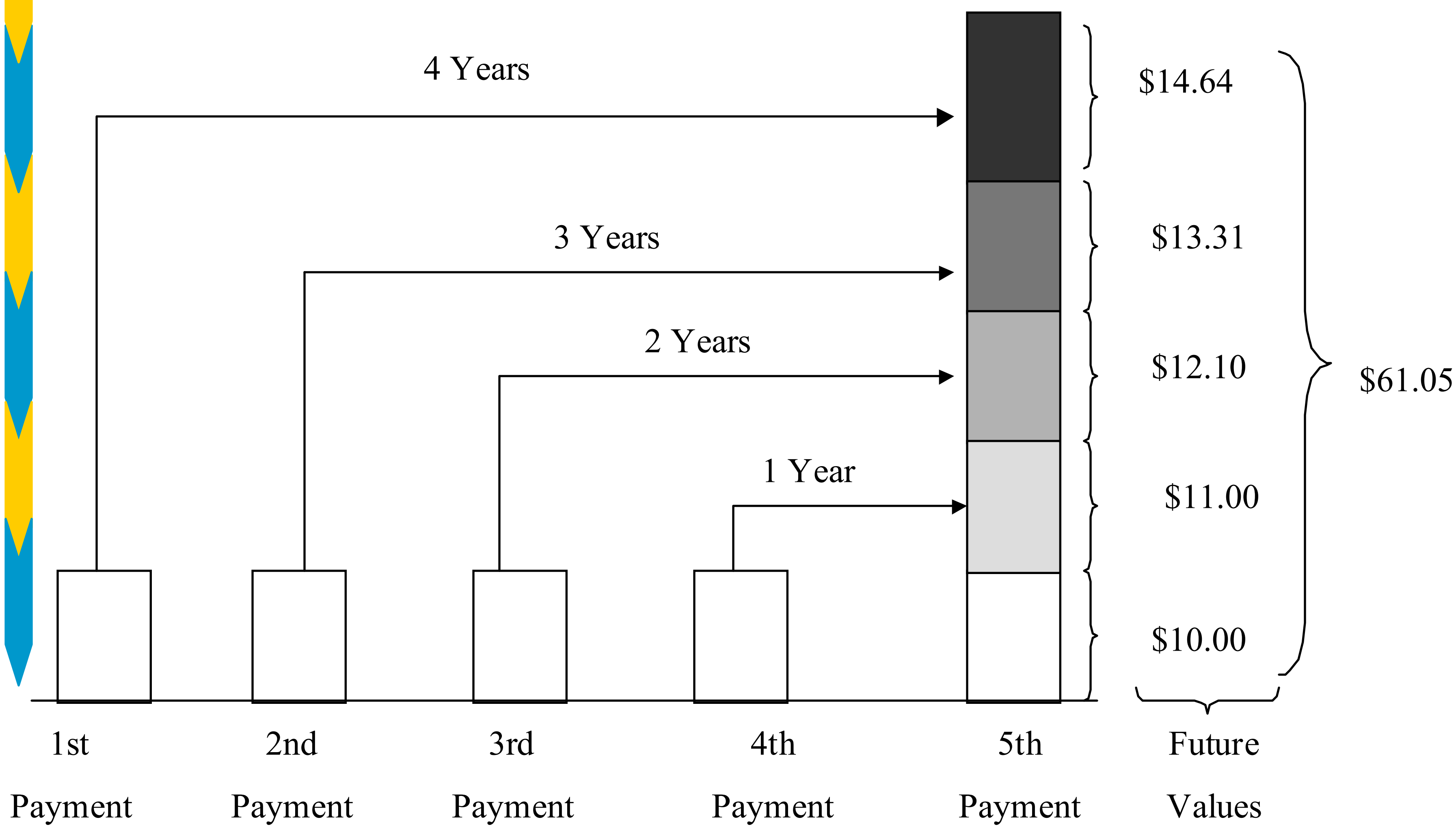
+ 1 = y^x 36=

- 1 =

/ rcl * 500 =

$18810.28

Figure 10.3
Contribution of Each Payment to an Annuity's Future Value



Suppose you decide to save $75/month for the next three years. If you invest all of these savings in an account which will pay you 7% pa compounded monthly, determine:

a) the total in the account after 3 years

b) the amount you deposited

c) the amount of interest earned

PMT =	- 75
I/Y =	7/12
n =	3*12 = 36
PV =	0
FV =	?

FV = **$2994.76**

Total deposits
= 75 * 36
= **$2700.00**

Interest earned
= 2994.76 - 2700
= **$294.76**

Financial Calculator sol'n:

7/12 = I/Y
36 n
0 PV
75 +/- PMT
comp FV

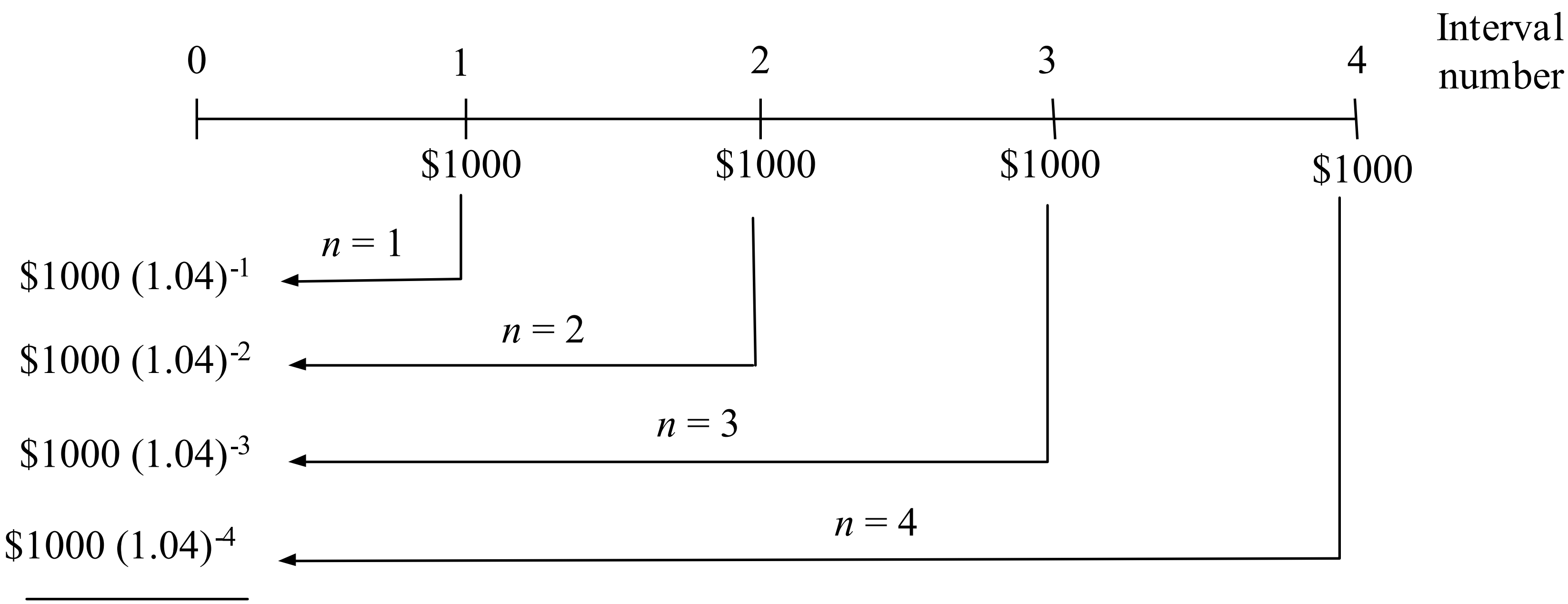
Present Value of an Ordinary Simple Annuity

the sum of the present values of all the payments

$$PV = PMT\left[\frac{1 - (1+i)^{-n}}{i}\right]$$

Figure 10.4

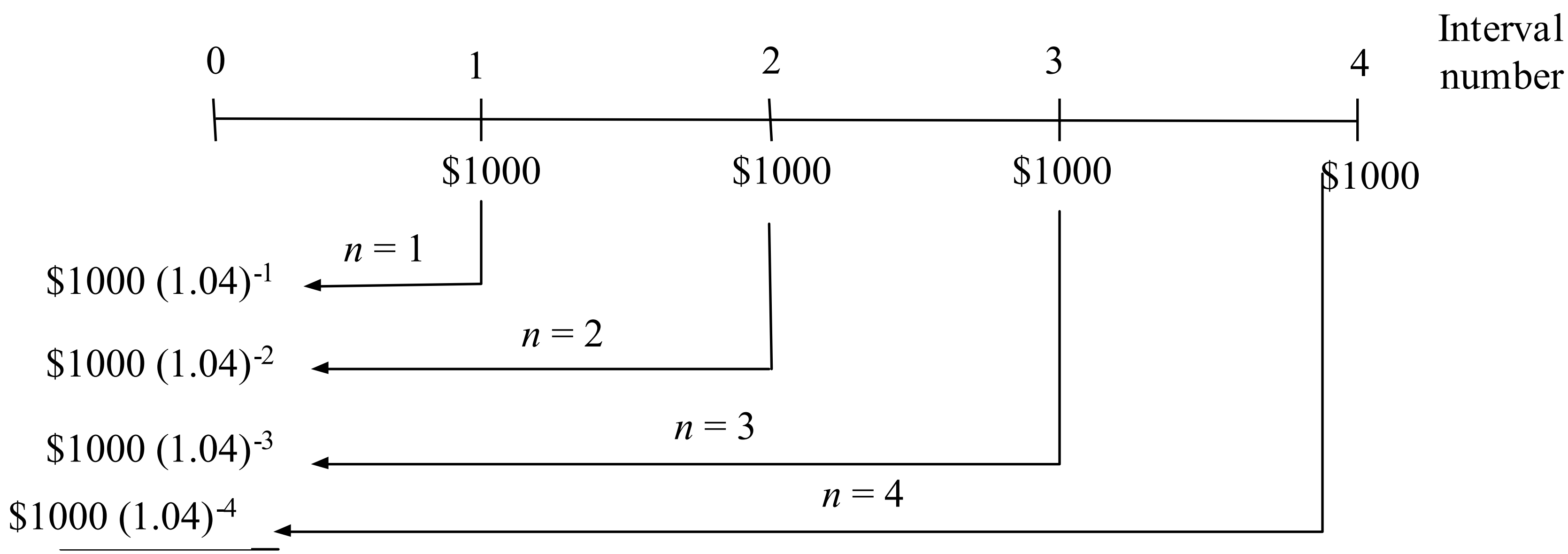
The Present Value of a Four-Payment Ordinary Simple Annuity



Sum = *PV* of annuity

Figure 10.4

The Present Value of an Ordinary Simple Annuity



Sum = *PV* of annuity

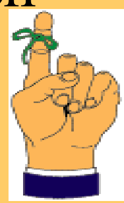
$$PV = \$1000\,(1.04)^{-1} + \$1000\,(1.04)^{-2} + \$1000\,(1.04)^{-3} + \$1000\,(1.04)^{-4}$$

$$= \$961.54 + \$924.56 + \$889.00 + \$854.80$$

$$= \$3629.90$$

You overhear your buddy saying the he is repaying a loan at $500 every month for the next four months. The interest rate he has been charged is 12%pa compounded monthly. Figure out the size of the loan, and the amount of interest involved.

Interest: use **12**, not **.12** when using financial calculator



n = 4 payments

Since you are making payments, not receiving them, PMT = **-**500

at the end of the loan, you don't owe any money, so FV = 0

PV = **$1950.98**

Interest charged =

2000 - 1950.98 = **$49.02**

Financial Calculator sol'n:

12/12=	I/Y
4	n
0	FV
500 +/-	PMT
comp	PV

You overhear your buddy saying the he is repaying a loan at $500 every month for the next four months. The interest rate he has been charged is 12%pa compounded monthly. Figure out the size of the loan, and the amount of interest involved.

Solving the same question using the formula:

$$PV = \frac{PMT[1 - (1+ i)^{-n}]}{i}$$

.12 / 12 = sto

+ 1 =

y^x 4 +/- =

- 1 = +/-

/ rcl * 500 =

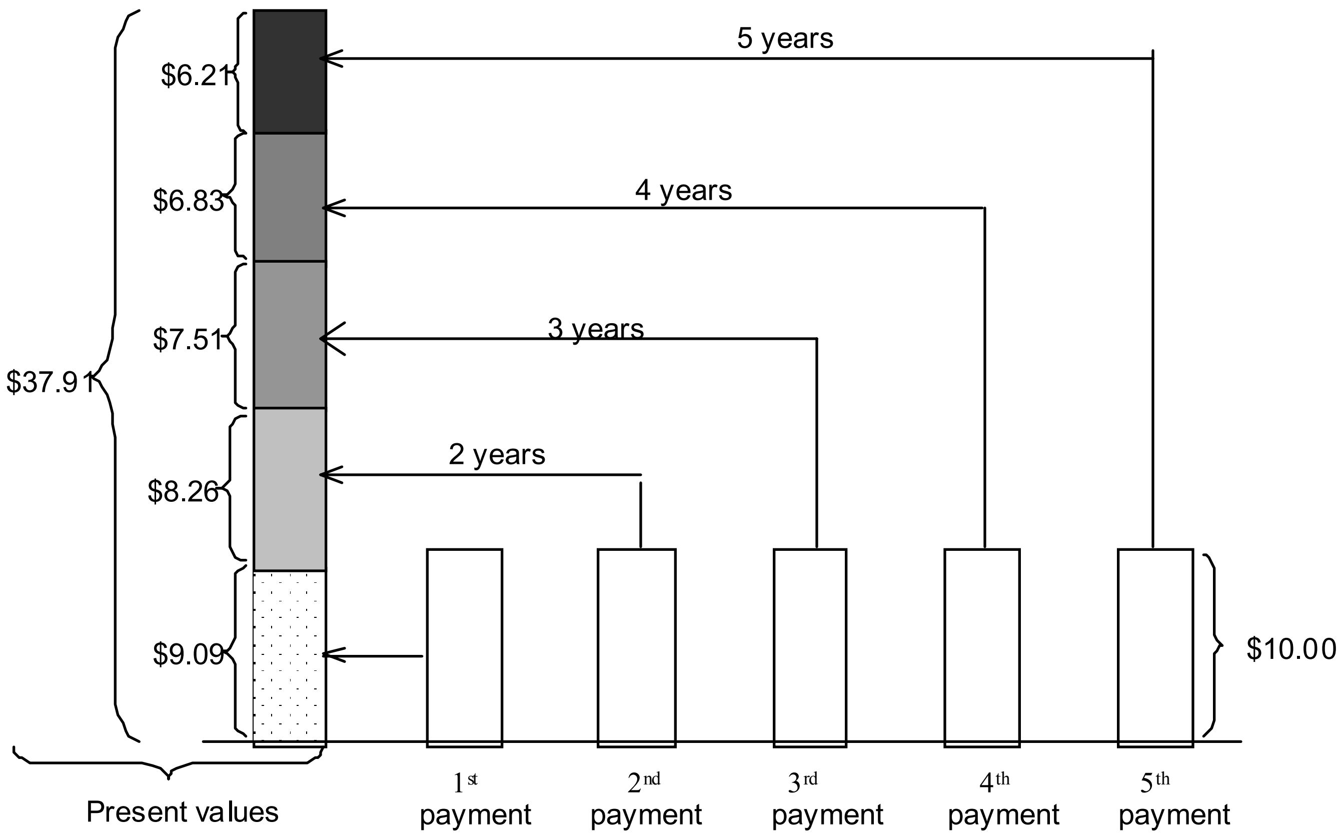
PV = **$1950.98**

Interest charged =

2000 - 1950.98 = **$49.02**

Figure 10.5

Contribution of Each Payment to an Annuity's Present Value



Deferred Annuities

A **deferred annuity** may be viewed as an *ordinary* annuity that does not begin until a time interval (named the **period of deferral**) has passed.

d = Number of payment intervals in the period of deferral

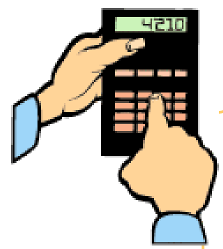
Two-step procedure to find PV:

> Calculate the present value, PV_1, of the payments at the end of the period of deferral—this is just the PV of an ordinary annuity.

> Calculate the present value, PV_2, of the step 1 amount at the beginning of the period of deferral.

If this same buddy doesn't begin to repay his loan for another 11 months, at a rate $500 every month for four months. The interest rate is still 12%pa compounded monthly. Determine the size of the loan.

Financial Calculator sol'n:



12/12= I/Y

4 n

0 FV

500 +/- PMT

comp PV

The deferral period is 10 months

PV = **$1950.98**

This is the value 10 months from now

Today's value:

1950.98 +/- FV

0 PMT

10 n

comp PV

$1766.20

General Annuities

- $$c = \frac{\textbf{number of compoundings per year}}{\textbf{number of payments per year}}$$
- Use $\boxed{i_2 = (1+i)^c - 1}$ to calculate the equivalent periodic rate that matches the payment interval.
- Use this equivalent periodic rate as the value for "*i*" in the appropriate simple annuity formula, or as the value entered into the **i** memory of the financial calculator.

Suppose you decide to save $50/month for the next three years. If you invest all of these savings in an account which will pay you 7% pa compounded semi-annually, determine the total in the account after 3 years.

Suppose you decide to save $50/month for the next three years. If you invest all of these savings in an account which will pay you 7% pa compounded semi-annually, determine the total in the account after 3 years.

Note the differing compounding frequency and payment intervals

This is a GENERAL annuity and so we need to calculate ***c***

$$c = \frac{\text{number of compoundings per year}}{\text{number of payments per year}}$$

$$\frac{2}{12}$$

= .16666

Store it!

Suppose you decide to save $50/month for the next three years. If you invest all of these savings in an account which will pay you 7% pa compounded semi-annually, determine the total in the account after 3 years.

PMT = -50

PV = 0

FV = ?

n = 3*12 = 36

i = .07/2

c = 2/12 = .16666

i_2 =

Once we have c, we need to determine i_2.

$i_2 = (1 + i)^c - 1$

2 / 12 = sto
.07 / 2 = + 1 =
y^x rcl =
- 1 = * 100 =

I/Y

4 n

0 FV

500 +/- PMT

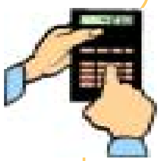
comp PV

FV = **$1993.51**

Tip: *Improving the Accuracy of Calculated Results*

- the value for ***c*** can be a ***repeating*** decimal
- when this happens, save *c* in memory
- your calculator then retains at least two more digits than you see in the display.
- when you need the exponent for the **y^x** function, recall the value for *c* from the memory
- the value for i_2 should be saved in memory as soon you calculate it. Whenever i_2 is needed in a subsequent calculation, recall it from the memory.

Reid David made annual deposits of $1,000 to Fleet Bank, which pays 6% interest compounded annually. After 4 years, Reid makes no more deposits. What will be the balance in the account 10 years after the last deposit?



6 I/Y

4 n

0 PV

1000 +/- PMT

comp FV

FV = **$4374.62**

Future value of an annuity

Future value of a lump sum

4374.62 +/- PV

10 n

0 PMT

comp FV

$7,834.03