The Nature of Mathematics

Definition

According to the various definitions, mathematics is the science of measurement, quality and magnitude. According to New English Dictionary, “Mathematics, in a strict sense, is the abstract science which investigates deductively the conclusions implicit in the elementary conceptions of spatial and numerical relations”. It has also been defined as the science of numbers and space. Its Hindi or Punjabi name is Ganita which means the science of calculation. It is a systemized, organized and exact branch of science.

Explanation

The term mathematics has been interpreted and explained in various ways. It is the numerical and calculation part of man’s life and knowledge. It helps the man to give the exact interpretation to his ideas and conclusions it deals with quantitative facts and relationships as well as with problems involving space and form. It also deals with relationships between magnitudes. It enables the man to study various phenomenon in space and establish various relationships between them. It explains that this science is a byproduct of our empirical knowledge. From our observations of physical and social environment, we form certain intuitive ideas or notions called postulates or axioms. By the process of reasoning, we move upwards and work out mathematical results at the abstract level. “Mathematics may also be defined as the science of abstract form. The discernment of structure is essential no less to the appreciation of a painting or a symphony that to understand the behavior of a physical system; no less in economics than in astronomy. Mathematics studies order abstracted from the particular objects and phenomena which exhibit it and in generalized form.

Science of Logical Reasoning

Mathematics is also called the science of logical reasoning. In it, we, approach everything with a question mark in our mind. As Locke has said, “Mathematics is a way to settle in the mind a habit of reasoning”. Here, the results are developed through a process of reasoning. There are few premises on which we base our reasoning. The conclusions follow naturally from the given facts when logical reasoning is applied to the same. The reasoning in mathematics is of peculiar kind and possess a number of characteristics such as simplicity, accuracy, certainty of results, originality similarity to the reasoning of life, and verification.

Inductive and Deductive Reasoning

Reasoning in mathematics is of two types;

i. **Inductive Reasoning**
ii. **Deductive Reasoning**

In the beginning, mathematics arises out of practical applications and it is mostly inductive and intuitive. “Mathematics in the beginning is not a deductive science; it is an inductive, experimental science and guessing is the experimental tool of mathematics. Mathematicians, like all other scientists, formulate their theories from bunches, analogies and simple examples. They are pretty confident that what they are trying to prove is correct and in writing these, they use only the bull dozer of logical deduction.

Inductive reasoning

When the statements or the prepositions are based on general observations and experience, reasoning is called inductive. When we can show that a particular property holds good in a sufficient number of cases, we conclude that it will also hold good in all similar cases. This type of reasoning is known as inductive reasoning.

Deductive Reasoning

Deductive reasoning is based on self-evident truths, postulates, axioms, etc. it precedes from a premises. Here the statements are the products of mind. This reasoning consist in comparing the statements and drawing a conclusion there from. Whitehead has correctly emphasized the place of deductive reasoning in mathematics by saying. “Mathematics in its widest sense is the development of all types of deductive reasoning”.

Essentials of Deductive System

Deductive reasoning in mathematics requires a number of essentials which have been discussed as follows.

Undefined Terms

In every branch of k knowledge, there are always certain terms which defy definitions. In their case, the definition if any fails to do justice to the concept concerned, the following three examples or Euclidean definitions will uphold the above statements:

Point: A point is that which has no part.

Line: A line is length without breadth

Straight Line: A straight line is that which lies evenly with the points on itself.

These definitions are far from clear in themselves. Modern mathematics has solved this difficulty by treating certain terms as undefined. Point, line and surface have been recognized as undefined terms in geometry. In algebra the terms set, number and variable are undefined.

Definitions

After taking some undefined terms we can now frame the definitions of technical terms. This we can do with the help of undefined terms, not technical language and other defined terms. For example in the case of triangle, “If A, B and C are three non collinear points, then triangle ABC is the union of the line segments AB, BC and AC”. Note that in this definition of triangle, we have use the term, collinear, point, union and line segment.

Postulates

As soon as we have collected a number of undefined and defined terms we are in a positions to make all the statements which we want to prove in the subject. Each statement to be proved is known as a propositions. While proving a propositions we base our arguments on the previously proved statements. But going back in the chain of propositions we reach the earliest ones prior to which no previously statement is available. For providing such initial propositions we have to depend upon some self-evident truths which are accepted as such without proof. These self-evident truths are known as postulates. These are assumed to be true without any necessity of explanations or proof.

Early Greeks tried to make distinction between postulates and axioms. The postulates were considered to be general truths common to all studies and the axioms as the truths relating to the special study at hand. Later on these concepts were modified. Postulates were considered as permissible constructions and all other initial assumptions were taken as axioms. However, in modern mathematics, no distinction is made between the two. The words postulates and the axioms are used synonymously.

Some of the postulates are given below.

1. A straight line may be drawn from any point to any other point.
2. A finite straight line may be produced to any length in that line.
3. A straight line has one and only one middle point.
4. Two n\straight line cannot intersect at more than one point.
5. All angles were equal.
6. A triangle has interior and exterior angles.
7. There is one and only one straight line through a point parallel to a given line.
8. A circle can be drawn with any point as center and with any length of radius.
9. Two circles intersect at two points.

Euclid regarded most of the other initial assumptions as axioms and called them common notion. These were accepted as true because of their conformity with common experience and sound judgment. The important notions are given below:

Things equal to the same thing, are equal to one another.

The whole is equal to the sum of its parts and is greater than part.

If $a>b$, $b>c$, then $a>c$.

Magnitudes of figures which can be made to coincide with one another are equal.

Mathematical Language and Symbolism

Another most important characteristics of mathematics which distinguishes it from many other subjects is its peculiar language and symbolism. Lindsay says, “Mathematics is the language of physical sciences and certainly no more marvelous language was ever created by the mind of the man”. Man has the ability to assign symbols for objects and ideas. Mathematical language and symbols cut short the lengthy statements and help the expressions of ideas or things in the exact form. Mathematical language is free from verbosity and helps in to the point, clear and exact information of facts. For example, instead of saying that the square of the sum of the two terms is equal to the sum of the square of the first term, square of the second term and double the product of the terms, we can simply write $(a + b)^2 = a^2 + b^2 + 2ab$ in symbolic form.

Mathematical results in their symbolic form help in solving numerous complicated problems. Most of the later progress in mathematics depends heavily on the learners’ ability to employ mathematics language and symbolism. It is reasonable to mention here, that most of the results of scientific inventions and discoveries are stated through mathematical language and symbolism.

Addition, subtraction, multiplication, division and equality are indicated by well known symbols. Some of the other important symbols are:

> For greater than

< For less than

< For angle

|| for parallel lines

≡ for congruency

√ for square root

E for summation

In fact, it is not possible to prepare a comprehensive and complete list of symbols used in mathematics. The students must be made familiar with them so that they are in a position to understand mathematical processes and conclusions and mathematical literature. Many of them loose interest in the subject because of their inability to understand mathematical language and symbolism. They cram the statements and processes and try to solve problems mechanically. Rather they should be enabled to understand and appreciate precision, brevity, logic, sharpness and beauty of mathematical language.

Pure and Applied Mathematics

Pure Mathematics

Pure mathematics involves systematic and deductive reasoning. It treats only theories and principles without regard to their application to concrete things. It is developed on an abstract, self-contained basis without any regard to any possible kind of practical applications that may follow. It consists of all those assertions as that if such and such proposition is true of anything, such and such another propositions is true of that thing.

Applied Mathematics

Applied mathematics is the applications of pure mathematics in the service of a given purpose. It has some direct or practical application to objects and happenings in the material world. It plays a great role in the development of various subjects. Every discovery in science owes much to applied mathematics. Principles of applied mathematics have been useful in the investigation of such phenomenon as heat, sound, light, optics, navigations and astronomy. Applied mathematics is a part of mathematics definitely related to or suggested by some tangible situations, though not always intended for practical use.it is the connecting link between pure mathematics on one side, physical, biological, social sciences and technology on the other. It acts and reacts not only on science technology but also on pure mathematics e.g., space dynamics ballistics, fluid dynamics, elasticity theory, theory of relativity, mathematical biology, and mathematical economics.

Relation between pure and Applied Mathematics

In the beginning man studied mathematical structure as found in his environment. By and by these turned into abstractions. For instance, take the case of notation system. Firstly man started counting with the help of concrete objects, then he developed number words for counting and this further led to the invention of abstract numerals 1, 2, 3… Very often certain structures are discovered in the social and physical sciences. The pure mathematicians try to develop some parallel structures in mathematics. They do not stop here. Rather they go on to discover more and more. Many theories and structures of pure mathematics have wide applications, which were not known at the time of their invention. For example, the theory of complex numbers was developed from the point of view of pure mathematics but now it finds intensive application in electricity, radio, related fields of physics and engineering. The reverse of this is also true. This is very well illustrated by Synge as follows:

1. A dive from the world of reality into the world of mathematics
2. A swim in the world of mathematics
3. A climb from the world of mathematics into the world of reality carrying the prediction in teeth.

Euclidean and Non-Euclidean Geometry

Euclidean Geometry

It goes after the name of Euclid, the famous Greek mathematician, and refers to the general type of geometry first logically organized by him. It is the type of geometry still studied universally in the high schools in the form of plane and solid geometry. Euclid introduced certain fundamental concepts as point, line, plane, angle, etc. He assumed certain self-evident truths which he called axioms and postulates. His geometry consists in proving geometrical truths with the help of axioms, postulates and previously proved truths. It employs a process of logical reasoning. Some of his definitions are given below:

1. Point: A point is that which has no part.
2. Line: A line is breathless length.
3. Straight Line: A straight line is a line which lies evenly with the points on itself.
4. Surface: A surface is that which has length and breadth only.
5. Plane Surface: A plane surface is a surface which lies evenly with the straight line on itself.
6. Angle: A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.

Some of his postulates are given below:

1. A straight line can be drawn from any point to any point.
2. A finite straight line can be produced continuously in a straight line.
3. A circle may be described with any center and any distance.
4. All right angles are equal to one another.

5. If a straight line falling on two straight line makes the interior angles on the same side less than two right angles, the two straight lines if produced indefinitely meet on that side on which are angles less than two right angles. (This is known as the Euclid postulate of parallels).
6. There is one and one straight line through a point parallel to a given straight line. (This follows the above postulate).

Shortcomings of Euclidean Geometry

It suffers from the following deficits:

1. Many concepts requiring definition or proof are over looked.
2. It makes use of certain assumptions which are not put as postulates. These assumptions are used in the proofs of certain theories.
3. Some of Euclid’s’ ideas are crude, e.g., he assumed that all angles must be less than two right angles.
4. Some of the postulates put forward are not true.
5. It does not take into consideration measurement on the spherical surface of the earth.
6. The practical applications of geometry are neglected.
7. It does not help the pupil in appreciating the basic characteristics of the perfect science of geometry.

Non Euclidean Geometry

The non-Euclidean geometry arose from the discussion, extending from the geek period to the present day. The starting point was the Euclid’s famous postulate of parallels which was proved false. Euclidean geometry came under the criticism from the renowned mathematicians like K.F. Guass, Lobachevslki and Reisman. Guass was first to prove the Euclid’s postulates of parallel false. Various other assumptions of Euclidean geometry were also criticized and disproved.

Non-Euclidean geometry takes into consideration the measurements on the spherical surface of the earth. This makes considerable difference in the concepts, terms and propositions of geometry. For example, a surveyor uses formulas of Euclidean geometry to measure the area of a plot or of a farm. Theoretically, he should use spherical geometry since his measurements are made on the spherical surface of the earth. But for a small area, the surface of the earth is almost plane.

In the non-Euclidean geometry, every straight line has a finite length equal to the length of half a great circle on the sphere. There can be an infinite number of straight lines through two points. The 180^0 and it approaches 180^0 when the area of the triangle approaches zero. Again in this geometry Pythagoras theorem is not true. It becomes true only when the area of the right angled triangle approaches zero. Unlike the postulates of the parallels, in this geometry there is no line through a point parallel to a given line.

Modern Mathematics

It will be interesting to bring in some topics and concepts of modern mathematics.

Sets

Set theory is one of the important aspects of modern mathematics. The study of the various sets with different structures is the corner stone of advanced mathematics. There is no doubt that the idea of set is basic to all mathematics.

A set is a collection of objects, also called aggregate. We speak of a set of the members of a team, set of instruments in a geometry box, set of items in a tea set is said to be defined when we know which objects form it i.e., we know definitely its elements. Some examples of defined sets are given below.

1. The set of integers 1,2,3,4, or A = (1, 2, 3, 4,)
2. Set of all positive integers or B= (1, 2, 3, 4…)
3. Set of all vowels or C= (a,e,I, o, u)
4. Set of prime numbers, or D = (2, 3, 5, 7, 11…)
5. Set of all rational numbers, etc.

Empty Set, Null Set, or Void Set

A set containing no element is called a void or empty or null set and is denoted by the symbole; i.e.; φ = (   )

Finite and Infinite Sets

A set containing only finite number of terms is known as a finite set. One which is not finite is called infinite set. For example, the set, (1, 2, 3, 4) is a finite set whereas the set of all natural numbers 1, 2, 3, 4… is an infinite set.

There are large numbers of terms in this area which need to be known and understood. Some of them are: equal sets, equivalent sets, sub-set, universal set, power set, complement of a set, intersection of sets, union of sets, disjoints sets.

Topology

Topology is derived from a Greek word which means a place. Previously it was defined as the study of situation or position. It was known as position analysis in which the shape and size of configuration are unimportant.

It has its roots in the 9th century in the researchers of Guass, Riesmann and Cautor but it made great advances with the advent of the 20th century.

It is mainly concerned with the intrinsic properties of figures i.e., the properties of figures themselves and not the properties concerning their relationship to any surrounding space in which they may be embedded. Topology is qualitative mathematics which deals with the intrinsic qualitative properties of objects or figures. These properties are independent of size, location and shape. Intrinsic qualitative property is the property that does not change even when the object under consideration is subjected to stretching and bending without tearing./ without tearing is important because the points originality close together remain close throughout the process of stretching . The object whose intrinsic properties are to be studied may be anything, a geometrical figure, rubber band, a collection of functions, an abstract space etc.

Topology is sometime known as rubber sheet geometry. If we draw certain figures on a rubber sheet and stretch it, the intrinsic properties of the figures is unaltered.

The object of topology is to study the properties of figures i.e., connectivity, orientibility. Et., that persists even when the figures are subjected to common deformation. If we imagine a circle made of pliable material, being subjected to continuous stretching and bending without being torn in the process, the circle is topologically unaltered. In topology, such figures, circles, triangles, ellipses, polygons are topologically equivalent because one figure can be transformed into the other without breaking or tearing, all such figures are known by the same term simple closed curve.

In topology, the figures can be stretched, shrunk, bent , distorted or folded in any way, so long as nearly points remain close to one another. The points which are in contact remain in contact and the points not in contact cannot some in contact. In topology, neither breaks nor fusion can arise. Thus the transformation of a circle into a square or a triangle is a topological transformation, but the torn and sphere are topologically distinct. One cannot be transformed into the other without breaking or tearing.

Topological studies

1. **Topological properties of figures**
2. **Topological transformation**
3. **Arbitrary continuous transformation of geometrical figures**

In topology, we consider set of points. A point may stand for an object, a figure, a book, a dog, or a point in the sense of Euclidean geometry. Topology is the study of intrinsic qualitative aspects of points. The kinds of sets are known a spaces in topology.

Algebraic System

In order to understand algebraic system, it is essential to have knowledge of sets, mapping and compositions. When a set is equipped with a binary compositions, it is known as an algebraic system. Some of the algebraic system are groups, rings, fields, vector spaces, etc. Let us take up these particular systems one by one:

Group

A set with composition is called a group, it satisfies the following conditions (adopting multiplication):

1. The composition is associative, i.e., (ab)c = a(bc) for every a, b, c belonging to G.
2. There is an identity e belonging to G such that ae = a = ea.

3. Every element of G has its inverse in G. That is for every a belonging to G, there corresponds an element b of G such that ab = e = ba

The set of all integers is a group. This set consists of integers…-3, -2, -1, 0, 1, 2, 3,…

Let us see if it satisfies the above conditions. First condition is obvious because we know the relations of the form.

(4 + 5) + 6 = 4 + (5 + 6)

Here + is the compositions as this corrects every ordered pair of numbers wih another number of the same set e.g., 4 + 5 = 9. Here + connects 4, 5 with 9.

As regards the second condition, see here 'O' is the identity.

4 + 0 = 0 + 4 or 5 + 0 = 0 +5

Also we know that -9 + 9 = 0 = 9 – 9

i.e., every element has the inverse. Hence the set of all integers forms a group.

The set of all natural numbers is not a group because it does not satisfy the third condition.

Ring

A set of R with two structures or binary compositions (adopting additive) and multiplicative notation) is a ring if it satisfies the following conditions:

1. The set R is a commutative group for assistive composition.

Commutative property a + b = b + a, for every a, b, belonging to R.

2. Multiplication is associative i.e., (ab) c = a (bc) for every a, b, c belonging to R.
3. Multiplication is right as well as left distributive i.e., a (b + c) = ab + ac (Right distribution)

i.e., (b + c) a = ba +ca (Left distribution)

All integers forms a ring because it is commutative group in respect to addition (Zero being the identity). Also the rational numbers and complex numbers form rings.

Field

A ring is called a field if it has at least two elements and

1. Is commutative
2. Has unity (identity for multiplication)
3. Has inverse for every non-zero element.

All rational numbers, all real numbers, all complex numbers form fields.