Karnaugh Map & Boolean Expression Simplification

Mapping a Standard POS Expression

For a Standard POS expression, a 0 is placed in the cell corresponding to the product term (maxterm) present in the expression. The cells are not filled with 0s have 1s. The Standard POS expression having a Domain of three variables $(A+B+\overline{C}).(A+\overline{B}+C).(\overline{A}+B+\overline{C}).(\overline{A}+\overline{B}+\overline{C})$ uses a 3-Variable Karnaugh Map. The sum terms or the Maxterms are 1, 2, 5 and 7. The expression can be represented by a K-Map by placing a 0 at Maxterm locations 1, 2, 5 and 7 and placing 1 at remaining places. Any of the two K-maps can be used. Figure 11.1.

AB\C	0	1
00	1	0
01	0	1
11	1	0
10	1	0

A\BC	00	01	11	10
0	1	0	1	0
1	1	0	0	1

Figure 11.1 Mapping a Standard POS expression

Karnaugh Map simplification of POS expressions

POS expressions can be easily simplified by use of the K-Map in a manner similar to the method adopted for simplifying SOP expressions. After the POS expression is mapped on the K-map, groups of 0s are marked instead of 1s based on the rules for forming groups used for simplifying SOP.

In the next step minimal sum terms are determined. Each group, including a group having a single cell, represents a sum term having variables that occur in only one form either complemented or un-complemented.

A 3-variable K-map yields

- A sum term of three variables for a group of 1 cell
- A sum term of two variables for a group of 2 cell
- A sum term of one variable for a group of 4 cell
- A group of 8 cells yields a value of 0 for the expression.

A 4-variable K-map yields

- A sum term of four variables for a group of 1 cell
- A sum term of three variables for a group of 2 cell
- A sum term of two variables for a group of 4 cell
- A sum term of one variable for a group of 8 cell
- A group of 16 cells yields a value of 0 for the expression.

Example 1 & 2

AB\C	0	1
00	0	1
01	1	0
11	1	1
10	0	1

A\BC	00	01	11	10
0	0	1	1	1
1	1	0	0	0

Figure 11.2 Simplification of POS expression using a 3-variable K-Map

A POS expression having 3 Maxterms is mapped to a 3-variable column based K-map. A single group of two cells and a group of one cell are formed.

- The first group of 0s comprising of cells 0 and 4 forms the sum term $(B+C)$
- The second group comprising of cell 3 forms the sum term $(A+\overline{B}+\overline{C})$

The three term POS expression simplifies to a 2 term POS expression $(B+C).(A+\overline{B}+\overline{C})$.

A POS expression having 4 Maxterms is mapped to a 3-variable column based K-map. Two groups of 2 cells each and a third group of single cell are formed.

- The single cell group comprising of cell 0 forms the sum term $(A+B+C)$
- The second group of 0s comprising of cells 5 and 7 forms the sum term $(\overline{A}+\overline{C})$
- The third group of 0s comprising of cells 6 and 7 forms the sum term $(\overline{A}+\overline{B})$

The four term POS expression simplifies to a 3 term POS expression $(A+B+C).(\overline{A}+\overline{C}).(\overline{A}+\overline{B})$.

Example 3 & 4

AB\C	0	1
00	0	0
01	1	1
11	1	1
10	0	1

A\BC	00	01	11	10
0	0	0	1	1
1	1	1	1	0

Figure 11.3 Simplification of POS expression using a 3-variable K-Map

A POS expression having 3 Maxterms is mapped to a 3-variable column based K-map. Two groups of two cells are formed.

- The first group of 0s comprising of cells 0 and 1 forms the sum term $(A+B)$
- The second group of 0s comprising of cells 0 and 4 forms the sum term $(B+C)$

The three term POS expression simplifies to a 2 terms POS expression $(A+B).(B+C)$

A POS expression having 3 Maxterms is mapped to a 3-variable column based K-map. One group of 2 cells and another group of single cell are formed.

- The first group of 0s comprising of cell 0 and 1 forms the sum term $(A + B)$
- The second group comprising of cell 6 forms the sum term $(\overline{A} + \overline{B} + C)$

The three term POS expression simplifies to a 2 term POS expression $(A + B).(\overline{A} + \overline{B} + C)$

Example 5

AB\CD	00	01	11	10
00	0	1	1	0
01	0	0	1	1
11	1	1	1	1
10	1	1	1	0

Figure 11.4 Simplification of POS expression using a 4-variable K-Map

A POS expression having 5 Maxterms is mapped to a 4-variable column based K-map. Three groups of two cells are formed.

- The first group of 0s comprising of cells 4 and 5 forms the sum term $(A + \overline{B} + C)$
- The second group of 0s comprising of cells 0 and 4 forms the sum term $(A + C + D)$
- The third group of 0s comprising of cells 2 and 10 forms the sum term $(B + \overline{C} + D)$

The five term POS expression has reduced to a 3 term POS expression $(A + \overline{B} + C).(A + C + D).(B + \overline{C} + D)$

Example 6

AB\CD	00	01	11	10
00	0	0	1	0
01	0	0	1	1
11	1	0	1	1
10	1	0	1	0

Figure 11.5 Simplification of POS expression using a 4-variable K-Map

A POS expression having 8 Maxterms is mapped to a 4-variable column based K-map. Two groups of 4 cells and one group of two cells are formed.

- The first group of 0s comprising of cells 0, 1, 4 and 5 forms the sum term $(A + C)$
- The second group of 0s comprising of cells 1, 5, 9 and 13 forms the sum term $(C + \overline{D})$
- The third group of 0s comprising of cells 2 and 10 forms the sum term $(B + \overline{C} + D)$

The eight term POS expression has reduced to a 3 term POS expression $(A + C).(C + \overline{D}).(B + \overline{C} + D)$.

Example 7

AB\CD	00	01	11	10
00	1	0	1	1
01	0	0	0	1
11	1	1	1	0
10	1	0	1	1

Figure 11.6 Simplification of POS expression using a 4-variable K-Map

A POS expression having 6 Maxterms is mapped to a 4-variable column based K-map. Three groups of 2 cells and one group of a single cell are formed.

- The first group of 0s comprising of cells 4 and 5 forms the sum term $(A + \overline{B} + C)$
- The second group of 0s comprising of cells 5 and 7 forms the sum term $(A + \overline{B} + \overline{D})$
- The third group of 0s comprising of cells 1 and 9 forms the sum term $(B + C + \overline{D})$
- The fourth group comprising of cell 14 forms the sum term $(\overline{A} + \overline{B} + \overline{C} + D)$

The six term POS expression has reduced to a 4 term POS expression $(A + \overline{B} + C).(A + \overline{B} + \overline{D}).(B + C + \overline{D}).(\overline{A} + \overline{B} + \overline{C} + D)$

Converting between POS and SOP using the K-map

Converting between the two forms of standard expressions is very simple. If the 1s mapped on the K-map are grouped together they form the product terms of the SOP expression. Similarly, if the 0s mapped on the K-map are grouped together they form the sum terms of the POS expression

Consider the POS expression $(A + \overline{B} + C).(A + \overline{B} + \overline{D}).(B + C + \overline{D}).(\overline{A} + \overline{B} + \overline{C} + D)$

AB\CD	00	01	11	10
00	1	0	1	1
01	0	0	0	1
11	1	1	1	0
10	1	0	1	1

AB\CD	00	01	11	10
00	1	0	1	1
01	0	0	0	1
11	1	1	1	0
10	1	0	1	1

Figure 11.7 Converting between SOP and POS using K-map

An equivalent SOP expression can be obtained by grouping the 1s together.

$\overline{B}\overline{D} + \overline{B}C + AB\overline{C} + ABD + \overline{A}C\overline{D}$

Five-Variable Karnaugh Map

A K-map for 5 variables can be constructed by using two 4-variable K-maps. Figure 11.8. The cells 0 to 15 lie in the 4-variable map A=0 and cells 16 to 31 lie in the 4-variable map A=1.

The two, 4-variable maps are considered to be lying on top of each other. Thus a two dimensional map is formed. Rules for grouping of 0s and 1s remain unchanged. In a 2-dimensional map, the groups of adjacent 0s or 1s can also span both the maps. In a 5-variable Karnaugh map groups of 2, 4, 8, 16 and 32 can be formed.

BC\DE	00	01	11	10
00	0	1	3	2
01	4	5	7	6
11	12	13	15	14
10	8	9	11	10

BC\DE	00	01	11	10
00	16	17	19	18
01	20	21	23	22
11	28	29	31	30
10	24	25	27	26

Figure 11.8 5-variable Karnaugh Map using A=0 and A=1 maps

Mapping, Grouping and Simplification using 5-variable Karnaugh maps is identical to those of 3 and 4 variable Karnaugh maps.

Simplification of 5-Variable Karnaugh Map

BC\DE	00	01	11	10
00	0	1	0	1
01	0	1	0	0
11	0	0	0	1
10	0	0	1	1

BC\DE	00	01	11	10
00	1	1	0	0
01	1	1	0	0
11	0	0	0	1
10	0	1	1	1

Figure 11.9 5-variable Karnaugh Map Simplification

The 5-variable Karnaugh map is mapped with Minterms in plane A=0 and A=1 respectively. Consider the groups that are formed.

- Starting with A=0 map. The cells 1 and 5 form a group of two cells. These two cells along with cells 17 and 21 in map A=1 from a group of 4 cells. This group of 4 cells represents the term $\overline{B}\overline{D}E$
- The cell 2 in map A=0. Cell 2 does not form a group with any adjacent cells. Therefore it is a group of single cell having the product term $\overline{A}\overline{B}\overline{C}D\overline{E}$
- The cells 10 and 11 in map A=0. These two cells form a group of four with adjacent cells 26 and 27 in map A=1. Therefore the group of 4 cells represents the product term $B\overline{C}D$

- Tthe cells 11 and 14 in map A=0 and cells 26 and 30 in map A=1represent a group of 4 cells representing the product term $BD\overline{E}$

Now considering the map A=1.

- The 4 cells 16, 17, 20 and 21 represent the product term $A\overline{B}\overline{D}$
- The cell 25 along with cell 27 in map A=1 represent the product term $AB\overline{C}E$

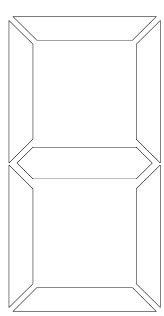
Functions having multiple outputs

In the discussions on Boolean expressions and Function Tables that represent Boolean functions it has been assumed that Logic Circuits have multiple inputs and single output. Practical Logic circuits however, have multiple inputs and multiple outputs. Circuits having a single output or multiple outputs are treated in the same manner.

Circuits having multiple outputs are represented by multiple function tables one for each output or a single function table having multiple output columns. The example of a BCD to 7-Segment Decoder circuit which has 4 inputs and 7 outputs is considered to explain functions having multiple outputs.

7-Segment Display

The 7-segment display digit is shown. Figure 11.10. 7-Segment Display is used to display the decimal numbers 0 to 9. A 7-segment display digit has 7 segments a, b, c, d, e, f and g that are turned on/off by a digital circuit depending upon the number that is to be displayed.



Digit	Segments
0	a, b, c, d, e, f
1	b, c
2	a, b, d, e, g
3	a, b, c, d, g
4	b, c, f, g
5	a, c , d, f, g
6	a, c, d, e, f, g
7	a, b, c
8	a, b, c, d, e, f, g
9	a, b, c, d, f, g

Figure 11.10 7-Segment Display

Different set of segments have to be turned on to display different digits. For example, to display the digit 3, segments a, b, c, d and g have to be turned on. To display the digit 7, segments a, b and c have to be turned on. The table indicates the segments that are turned on for each digit.

The circuit that turns on the appropriate segments to display a digit is known as a BCD to 7-Sement Decoder. The input to the BCD to 7-Segment decoder circuit is a 4-bit BCD number between 0 and 9. The seven output lines of the decoder connect to the 7 segments. Figure 11.11.

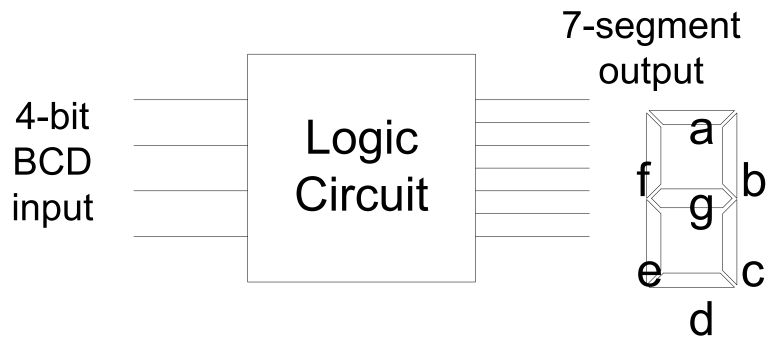


Figure 11.11 BCD to 7-Segment Decoder

To implement the decoder circuit having 4 inputs and 7 outputs, function tables have to be drawn which represent the output status of each output line for all combinations of inputs. For example, the segment a is turned on when the 4-bit input is 0, 2, 3, 5, 6, 7, 8 and 9. Similarly, the segment b is turned on for 0, 2, 3, 4, 7, 8 and 9 combinations of inputs. Thus seven expressions, one for each segment has to be be determined before the decoder circuit can be implemented.

Seven function tables are required to represent the input/output combinations for each segment. The seven function tables for segments a, b, c, d, e, f and g are shown. Figure 11.12a-g. To determine the seven expressions for each of the seven outputs, seven 4-variable Karnaugh maps are used. The Karnaugh maps and the simplified expressions are shown. Figure 11.13a-g. An alternate way of representing the seven Function tables is to have a single function table with the four columns representing the 4-bit input BCD number and seven output columns each representing one of the seven segments a, b, c, d, e, f and g respectively.

Since the 4-bit input to the decoder circuit can have 16 possible input combinations, therefore each of the seven Function tables have sixteen input combinations. However, the last 6 input combinations are don't care as these combinations never occur because the input to the circuit is a 4-bit BCD number. The don't care states help in simplifying the Boolean expressions for the seven segments.

Input				Output	Input				Output
A	B	C	D	Seg. a	A	B	C	D	Seg. a
0	0	0	0	1	1	0	0	0	1
0	0	0	1	0	1	0	0	1	1
0	0	1	0	1	1	0	1	0	x
0	0	1	1	1	1	0	1	1	x
0	1	0	0	0	1	1	0	0	x
0	1	0	1	1	1	1	0	1	x
0	1	1	0	1	1	1	1	0	x
0	1	1	1	1	1	1	1	1	x

Figure 11.12a Function Table for Segment a

Input				Output	Input				Output
A	B	C	D	Seg. b	A	B	C	D	Seg. b
0	0	0	0	1	1	0	0	0	1
0	0	0	1	1	1	0	0	1	1
0	0	1	0	1	1	0	1	0	x
0	0	1	1	1	1	0	1	1	x
0	1	0	0	1	1	1	0	0	x
0	1	0	1	0	1	1	0	1	x
0	1	1	0	0	1	1	1	0	x
0	1	1	1	1	1	1	1	1	x

Figure 11.12b Function Table for Segment b

Input				Output	Input				Output
A	B	C	D	Seg. c	A	B	C	D	Seg. c
0	0	0	0	1	1	0	0	0	1
0	0	0	1	1	1	0	0	1	1
0	0	1	0	0	1	0	1	0	X
0	0	1	1	1	1	0	1	1	X
0	1	0	0	1	1	1	0	0	X
0	1	0	1	1	1	1	0	1	x
0	1	1	0	1	1	1	1	0	x
0	1	1	1	1	1	1	1	1	x

Figure 11.12c Function Table for Segment c

Input				Output	Input				Output
A	B	C	D	d	A	B	C	D	d
0	0	0	0	1	1	0	0	0	1
0	0	0	1	0	1	0	0	1	1
0	0	1	0	1	1	0	1	0	x
0	0	1	1	1	1	0	1	1	x
0	1	0	0	0	1	1	0	0	x
0	1	0	1	1	1	1	0	1	x
0	1	1	0	1	1	1	1	0	x
0	1	1	1	0	1	1	1	1	x

Figure 11.12d Function Table for Segment d

Input				Output	Input				Output
A	B	C	D	Seg. e	A	B	C	D	Seg. e
0	0	0	0	1	1	0	0	0	1
0	0	0	1	0	1	0	0	1	0
0	0	1	0	1	1	0	1	0	x
0	0	1	1	0	1	0	1	1	x
0	1	0	0	0	1	1	0	0	x
0	1	0	1	0	1	1	0	1	x
0	1	1	0	1	1	1	1	0	x
0	1	1	1	0	1	1	1	1	x

Figure 11.12e Function Table for Segment e

Input				Output	Input				Output
A	B	C	D	Seg. f	A	B	C	D	Seg. f
0	0	0	0	1	1	0	0	0	1
0	0	0	1	0	1	0	0	1	1
0	0	1	0	0	1	0	1	0	x
0	0	1	1	0	1	0	1	1	x
0	1	0	0	1	1	1	0	0	x
0	1	0	1	1	1	1	0	1	x
0	1	1	0	1	1	1	1	0	x
0	1	1	1	0	1	1	1	1	x

Figure 11.12f Function Table for Segment f

Input				Output	Input				Output
A	B	C	D	Seg. g	A	B	C	D	Seg. g
0	0	0	0	0	1	0	0	0	1
0	0	0	1	0	1	0	0	1	1
0	0	1	0	1	1	0	1	0	x
0	0	1	1	1	1	0	1	1	x
0	1	0	0	1	1	1	0	0	x
0	1	0	1	1	1	1	0	1	x
0	1	1	0	1	1	1	1	0	x
0	1	1	1	0	1	1	1	1	x

Figure 11.12g Function Table for Segment g

AB\CD	00	01	11	10
00	1	0	1	1
01	0	1	1	1
11	x	x	x	X
10	1	1	x	x

$a = A + C + BD + \overline{B}\overline{D}$

AB\CD	00	01	11	10
00	1	1	1	1
01	1	0	1	0
11	x	x	x	x
10	1	1	x	x

$b = \overline{B} + \overline{C}\overline{D} + CD$

AB\CD	00	01	11	10
00	1	1	1	0
01	1	1	1	1
11	x	x	x	x
10	1	1	x	x

$c = \overline{C} + D + B$

AB\CD	00	01	11	10
00	1	0	1	1
01	0	1	0	1
11	x	x	x	x
10	1	1	x	x

$d = A + \overline{B}\overline{D} + \overline{B}C + C\overline{D} + B\overline{C}D$

AB\CD	00	01	11	10
00	1	0	0	1
01	0	0	0	1
11	x	x	x	x
10	1	0	x	x

$e = \overline{B}\overline{D} + C\overline{D}$

AB\CD	00	01	11	10
00	1	0	0	0
01	1	1	0	1
11	x	x	x	x
10	1	1	x	x

$f = B + \overline{C}\overline{D} + B\overline{C} + B\overline{D}$

AB\CD	00	01	11	10
00	0	0	1	1
01	1	1	0	1
11	x	x	x	x
10	1	1	x	x

$$g = A + B\overline{C} + C\overline{D} + \overline{B}C$$

Figure 11.13a-g Karnaugh Maps and Simplified Boolean Expressions for Display Segments a to g