LECTURE # 3
Laws of Logic

APPLYING LAWS OF LOGIC
Using law of logic, simplify the statement form
\[ p \lor [\neg (\neg p \land q)] \]

Solution:
\[ p \lor [\neg (\neg p \land q)] \equiv p \lor [\neg (\neg p) \lor (\neg q)] \]
\[ \equiv p \lor [p \lor (\neg q)] \]
\[ \equiv [p \lor p] \lor (\neg q) \]
\[ \equiv p \lor (\neg q) \]

Which is the simplified statement form.

EXAMPLE Using Laws of Logic, verify the logical equivalence
\[ \neg (\neg p \land q) \land (p \lor q) \equiv p \]
\[ \neg (\neg p \land q) \land (p \lor q) \equiv (\neg (\neg p) \lor \neg q) \land (p \lor q) \]
\[ \equiv (p \lor \neg q) \land (p \lor q) \]
\[ \equiv p \lor (\neg q \land q) \]
\[ \equiv p \lor c \]
\[ \equiv p \]

SIMPLIFYING A STATEMENT:
“You will get an A if you are hardworking and the sun shines, or you are hardworking and it rains.”

Rephrase the condition more simply.

Solution:
Let \( p = \) “You are hardworking”
\( q = \) “The sun shines”
\( r = \) “It rains” . The condition is then \((p \land q) \lor (p \land r)\)

And using distributive law in reverse,
\[(p \land q) \lor (p \land r) \equiv p \land (q \lor r)\]

Putting \( p \land (q \lor r) \) back into English, we can rephrase the given sentence as

“You will get an A if you are hardworking and the sun shines or it rains.

EXERCISE:
Use Logical Equivalence to rewrite each of the following sentences more simply.

1. It is not true that I am tired and you are smart.
   {I am not tired or you are not smart.}

2. It is not true that I am tired or you are smart.
   {I am not tired and you are not smart.}

3. I forgot my pen or my bag and I forgot my pen or my glasses.
   {I forgot my pen or I forgot my bag and glasses.}

4. It is raining and I have forgotten my umbrella, or it is raining and I have forgotten my hat.
   {It is raining and I have forgotten my umbrella or my hat.}

CONDITIONAL STATEMENTS:
Introduction
Consider the statement:
"If you earn an A in Math, then I'll buy you a computer."
This statement is made up of two simpler statements:
p: "You earn an A in Math," and
q: "I will buy you a computer."
The original statement is then saying:
if \( p \) is true, then \( q \) is true, or, more simply, if \( p \), then \( q \).

We can also phrase this as \( p \) implies \( q \), and we write \( p \rightarrow q \).

**CONDITIONAL STATEMENTS OR IMPLICATIONS:**

If \( p \) and \( q \) are statement variables, the conditional of \( q \) by \( p \) is “If \( p \) then \( q \)” or “\( p \) implies \( q \)” and is denoted \( p \rightarrow q \).

It is false when \( p \) is true and \( q \) is false; otherwise it is true. The arrow ”\( \rightarrow \)” is the conditional operator, and in \( p \rightarrow q \) the statement \( p \) is called the hypothesis (or antecedent) and \( q \) is called the conclusion (or consequent).

**TRUTH TABLE:**

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**PRACTICE WITH CONDITIONAL STATEMENTS:**

Determine the truth value of each of the following conditional statements:

1. “If 1 = 1, then 3 = 3.” \( \text{TRUE} \)
2. “If 1 = 1, then 2 = 3.” \( \text{FALSE} \)
3. “If 1 = 0, then 3 = 3.” \( \text{TRUE} \)
4. “If 1 = 2, then 2 = 3.” \( \text{TRUE} \)
5. “If 1 = 1, then 1 = 2 and 2 = 3.” \( \text{FALSE} \)
6. “If 1 = 3 or 1 = 2 then 3 = 3.” \( \text{TRUE} \)

**ALTERNATIVE WAYS OF EXPRESSING IMPLICATIONS:**

The implication \( p \rightarrow q \) could be expressed in many alternative ways as:

- “if \( p \) then \( q \)”
- “\( p \) implies \( q \)”
- “\( p \) only if \( q \)”
- “\( p \) is sufficient for \( q \)”
- “\( q \) follows from \( p \)”
- “\( q \) if \( p \)”
- “\( q \) whenever \( p \)”
- “\( q \) is necessary for \( p \)”

**EXERCISE:**

Write the following statements in the form “if \( p \), then \( q \)” in English.

a) *Your guarantee is good only if you bought your CD less than 90 days ago.*

If your guarantee is good, then you must have bought your CD player less than 90 days ago.

b) *To get tenure as a professor, it is sufficient to be world-famous.*

If you are world-famous, then you will get tenure as a professor.

c) *That you get the job implies that you have the best credentials.*

If you get the job, then you have the best credentials.

d) *It is necessary to walk 8 miles to get to the top of the Peak.*

If you get to the top of the peak, then you must have walked 8 miles.

**TRANSLATING ENGLISH SENTENCES TO SYMBOLS:**

Let \( p \) and \( q \) be propositions:

- \( p = \) “you get an A on the final exam”
- \( q = \) “you do every exercise in this book”
r = “you get an A in this class”
Write the following propositions using p, q, and r and logical connectives.
1. To get an A in this class it is necessary for you to get an A on the final.
   SOLUTION  \( p \rightarrow r \)
2. You do every exercise in this book; You get an A on the final, implies, you get an A in the class.
   SOLUTION  \( p \land q \rightarrow r \)
3. Getting an A on the final and doing every exercise in this book is sufficient for getting an A in this class.
   SOLUTION  \( p \land q \rightarrow r \)

TRANSLATING SYMBOLIC PROPOSITIONS TO ENGLISH:
Let \( p \), \( q \), and \( r \) be the propositions:
\( p \) = “you have the flu”
\( q \) = “you miss the final exam”
\( r \) = “you pass the course”
Express the following propositions as an English sentence.
1. \( p \rightarrow q \)
   If you have flu, then you will miss the final exam.
2. \( \sim q \rightarrow r \)
   If you don’t miss the final exam, you will pass the course.
3. \( \sim p \land \sim q \rightarrow r \)
   If you neither have flu nor miss the final exam, then you will pass the course.

HIERARCHY OF OPERATIONS
FOR LOGICAL CONNECTIVES
\(~\) (negation)
\( \land \) (conjunction), \( \lor \) (disjunction)
\( \rightarrow \) (conditional)
Construct a truth table for the statement form \( p \lor \sim q \rightarrow \sim p \)

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Construct a truth table for the statement form \((p \rightarrow q) \land (\neg p \rightarrow r)\):

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**LOGICAL EQUIVALENCE INVOLVING IMPLICATION**

Use truth table to show \(p \rightarrow q \equiv \neg q \rightarrow \neg p\):

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same truth values
Hence the given two expressions are equivalent.

**IMPLICATION LAW**

\[ p \rightarrow q \equiv \neg p \lor q \]

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same truth values

**NEGATION OF A CONDITIONAL STATEMENT:**

Since \( p \rightarrow q \equiv \neg p \lor q \) therefore

\[ \neg (p \rightarrow q) \equiv \neg (\neg p \lor q) \]

\[ \equiv (\neg p) \land (\neg q) \text{ by De Morgan’s law} \]

\[ \equiv p \land \neg q \text{ by the Double Negative law} \]

Thus the negation of “if \( p \) then \( q \)” is logically equivalent to “\( p \) and not \( q \)”. Accordingly, the negation of an if-then statement does not start with the word if.

**EXAMPLES**

Write negations of each of the following statements:
1. If Ali lives in Pakistan then he lives in Lahore.
2. If my car is in the repair shop, then I cannot get to class.
3. If \( x \) is prime then \( x \) is odd or \( x \) is 2.
4. If \( n \) is divisible by 6, then \( n \) is divisible by 2 and \( n \) is divisible by 3.

**SOLUTIONS:**
1. Ali lives in Pakistan and he does not live in Lahore.
2. My car is in the repair shop and I can get to class.
3. \( x \) is prime but \( x \) is not odd and \( x \) is not 2.
4. \( n \) is divisible by 6 but \( n \) is not divisible by 2 or by 3.

**INVERSE OF A CONDITIONAL STATEMENT:**

The inverse of the conditional statement \( p \rightarrow q \) is \( \neg p \rightarrow \neg q \)

A conditional and its inverse are not equivalent as could be seen from the truth table.
different truth values in rows 2 and 3

**WRITING INVERSE:**
1. *If today is Friday, then \(2 + 3 = 5\).*
   - If today is not Friday, then \(2 + 3 \neq 5\).
2. *If it snows today, I will ski tomorrow.*
   - If it does not snow today I will not ski tomorrow.
3. *If \(P\) is a square, then \(P\) is a rectangle.*
   - If \(P\) is not a square then \(P\) is not a rectangle.
4. *If my car is in the repair shop, then I cannot get to class.*
   - If my car is not in the repair shop, then I shall get to the class.

**CONVERSE OF A CONDITIONAL STATEMENT:**
The converse of the conditional statement \(p \rightarrow q\) is \(q \rightarrow p\)
A conditional and its converse are not equivalent.

i.e., \(\rightarrow\) is not a commutative operator.

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**WRITING CONVERSE:**
1. *If today is Friday, then \(2 + 3 = 5\).*
   - If \(2 + 3 \neq 5\), then today is not Friday.
2. *If it snows today, I will ski tomorrow.*
   - I will ski tomorrow only if it snows today.
3. *If \(P\) is a square, then \(P\) is a rectangle.*
   - If \(P\) is a rectangle then \(P\) is a square.
4. *If my car is in the repair shop, then I cannot get to class.*
   - If I cannot get to the class, then my car is in the repair shop.

**CONTRAPOSITIVE OF A CONDITIONAL STATEMENT:**
The contrapositive of the conditional statement \(p \rightarrow q\) is \(\sim q \rightarrow \sim p\)
A conditional and its contrapositive are equivalent. Symbolically, \(p \rightarrow q \equiv \sim q \rightarrow \sim p\)

1. *If today is Friday, then \(2 + 3 = 5\).*
   - If \(2 + 3 \neq 5\), then today is not Friday.
2. *If it snows today, I will ski tomorrow.*
   - I will not ski tomorrow only if it does not snow today.
3. *If \(P\) is a square, then \(P\) is a rectangle.*
   - If \(P\) is not a rectangle then \(P\) is not a square.
4. *If my car is in the repair shop, then I cannot get to class.*
   - If I get to the class, then my car is not in the repair shop.