Combinations and Permutations

What's the Difference?

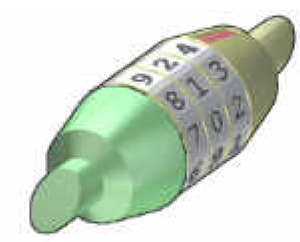
In English we use the word "combination" loosely, without thinking if the **order** of things is important. In other words:

"My fruit salad is a combination of apples, grapes and bananas" We don't care what order the fruits are in, they could also be "bananas, grapes and apples" or "grapes, apples and bananas", its the same fruit salad.

"The combination to the safe was 472". Now we **do** care about the order. "724" would not work, nor would "247". It has to be exactly **4-7-2**.

So, in Mathematics we use more *precise* language:

- If the order doesn't matter, it is a **Combination**.
- If the order **does** matter it is a **Permutation**.



So, we should really call this a "Permutation Lock"!

In other words:

A Permutation is an **ordered** Combination.

To help you to remember, think "**P**ermutation ... **P**osition"

Permutations

There are basically two types of permutation:

1. **Repetition is Allowed**: such as the lock above. It could be "333".
2. **No Repetition**: for example the first three people in a running race. You can't be first *and* second.

1. Permutations with Repetition

These are the easiest to calculate.

When we have ***n*** things to choose from ... we have ***n*** choices each time!

When choosing ***r*** of them, the permutations are:

$$\mathbf{n \times n \times \ldots} \ \textbf{\textit{(r times)}}$$

(In other words, there are **n** possibilities for the first choice, THEN there are **n** possibilites for the second choice, and so on, multplying each time.)

Which is easier to write down using an exponent of **r**:

$$\mathbf{n \times n \times \ldots (r\ times) = n^r}$$

Example: in the lock above, there are 10 numbers to choose from (0,1,...9) and we choose 3 of them:

$$\mathbf{10 \times 10 \times \ldots (3\ times) = 10^3 = 1{,}000\ permutations}$$

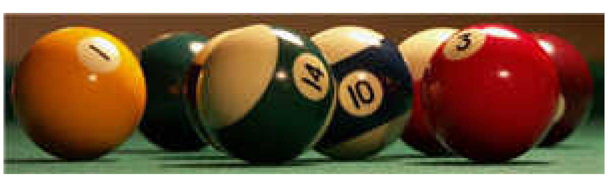
So, the formula is simply:

$$\mathbf{n^r}$$

where ***n*** is the number of things to choose from, and we choose ***r*** of them
(Repetition allowed, order matters)

2. Permutations without Repetition

In this case, we have to **reduce** the number of available choices each time.



For example, what order could 16 pool balls be in?

After choosing, say, number "14" we can't choose it again.

So, our first choice would have 16 possibilites, and our next choice would then have 15 possibilities, then 14, 13, etc. And the total permutations would be:

$$16 \times 15 \times 14 \times 13 \times \ldots = 20{,}922{,}789{,}888{,}000$$

But maybe we don't want to choose them all, just 3 of them, so that would be only:

$$16 \times 15 \times 14 = 3{,}360$$

In other words, there are 3,360 different ways that 3 pool balls could be selected out of 16 balls.

But how do we write that mathematically? Answer: we use the "factorial function"

The **factorial function** (symbol: **!**) just means to multiply a series of descending natural numbers. Examples:

- $4! = 4 \times 3 \times 2 \times 1 = 24$
- $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5{,}040$
- $1! = 1$

Note: it is generally agreed that **0! = 1**. It may seem funny that multiplying no numbers together gets us 1, but it helps simplify a lot of equations.

So, if we wanted to select **all** of the billiard balls the permutations would be:

$$16! = 20{,}922{,}789{,}888{,}000$$

But if we wanted to select just 3, then we have to stop the multiplying after 14. How do we do that? There is a neat trick ... we divide by **13!** ...

$$\frac{16 \times 15 \times 14 \times 13 \times 12 \ldots}{13 \times 12 \ldots} = 16 \times 15 \times 14 = 3{,}360$$

Do you see? **16! / 13! = 16 × 15 × 14**

The formula is written:

$$\frac{n!}{(n-r)!}$$

where ***n*** is the number of things to choose from, and we choose ***r*** of them
(No repetition, order matters)

Examples:

Our "order of 3 out of 16 pool balls example" would be:

$$\frac{16!}{(16-3)!} = \frac{16!}{13!} = \frac{20{,}922{,}789{,}888{,}000}{6{,}227{,}020{,}800} = 3{,}360$$

(which is just the same as: **16 × 15 × 14 = 3,360**)

How many ways can first and second place be awarded to 10 people?

$$\frac{10!}{(10-2)!} = \frac{10!}{8!} = \frac{3{,}628{,}800}{40{,}320} = 90$$

(which is just the same as: **10 × 9 = 90**)

Notation

Instead of writing the whole formula, people use different notations such as these:

$$P(n,r) = {}^nP_r = {}_nP_r = \frac{n!}{(n-r)!}$$

Example: ***P*(10,2) = 90**

Combinations

There are also two types of combinations (remember the order does **not** matter now):

1. **Repetition is Allowed**: such as coins in your pocket (5,5,5,10,10)
2. **No Repetition**: such as lottery numbers (2,14,15,27,30,33)

1. Combinations with Repetition

Actually, these are the hardest to explain, so we will come back to this later.

2. Combinations without Repetition

This is how lotteries work. The numbers are drawn one at a time, and if we have the lucky numbers (no matter what order) we win!

The easiest way to explain it is to:

- assume that the order does matter (ie permutations),
- then alter it so the order does **not** matter.

Going back to our pool ball example, let's say we just want to know which 3 pool balls were chosen, not the order.

We already know that 3 out of 16 gave us 3,360 permutations.

But many of those will be the same to us now, because we don't care what order!

For example, let us say balls 1, 2 and 3 were chosen. These are the possibilites:

Order does matter	Order doesn't matter
1 2 3 1 3 2 2 1 3 2 3 1 3 1 2 3 2 1	1 2 3

So, the permutations will have 6 times as many possibilites.

In fact there is an easy way to work out how many ways "1 2 3" could be placed in order, and we have already talked about it. The answer is:

3! = 3 × 2 × 1 = 6

(Another example: 4 things can be placed in **4! = 4 × 3 × 2 × 1 = 24** different ways, try it for yourself!)

So we adjust our permutations formula to **reduce it** by how many ways the objects could be in order (because we aren't interested in their order any more):

$$\frac{n!}{(n-r)!} \times \frac{1}{r!} = \frac{n!}{r!(n-r)!}$$

That formula is so important it is often just written in big parentheses like this:

$$\frac{n!}{r!(n-r)!} = \binom{n}{r}$$

where ***n*** is the number of things to choose from, and we choose ***r*** of them
(No repetition, order doesn't matter)

It is often called "n choose r" (such as "16 choose 3")

And is also known as the "Binomial Coefficient"

Notation

As well as the "big parentheses", people also use these notations:

$$C(n,r) = {}^nC_r = {}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Example

So, our pool ball example (now without order) is:

$$\frac{\mathbf{16!}}{\mathbf{3!(16\text{-}3)!}} = \frac{\mathbf{16!}}{\mathbf{3!\times 13!}} = \frac{\mathbf{20{,}922{,}789{,}888{,}000}}{\mathbf{6\times 6{,}227{,}020{,}800}} \quad \mathbf{= 560}$$

Or we could do it this way:

$$\frac{\mathbf{16\times 15\times 14}}{\mathbf{3\times 2\times 1}} = \frac{\mathbf{3360}}{\mathbf{6}} \quad \mathbf{= 560}$$

So remember, do the permutation, then reduce by a further "**r!**"

... or better still ...

Remember the Formula!

It is interesting to also note how this formula is nice and **symmetrical**:

$$\frac{n!}{r!(n-r)!} = \binom{n}{r} = \binom{n}{n-r}$$

In other words choosing 3 balls out of 16, or choosing 13 balls out of 16 have the same number of combinations.

$$\frac{\mathbf{16!}}{\mathbf{3!(16\text{-}3)!}} = \frac{\mathbf{16!}}{\mathbf{13!(16\text{-}13)!}} = \frac{\mathbf{16!}}{\mathbf{3!\times 13!}} \quad \mathbf{= 560}$$

Pascal's Triangle

We can also use Pascal's Triangle to find the values. Go down to row "n" (the top row is 0), and then along "r" places and the value there is our answer. Here is an extract showing row 16:

1 14 91 364 ...

1 15 105 455 1365 ...

1 16 120 **560** 1820 4368 ...

1. Combinations with Repetition

OK, now we can tackle this one ...

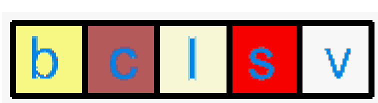
Let us say there are five flavors of icecream: **banana, chocolate, lemon, strawberry and vanilla**. We can have three scoops. How many variations will there be?

Let's use letters for the flavors: {b, c, l, s, v}. Example selections would be

- {c, c, c} (3 scoops of chocolate)
- {b, l, v} (one each of banana, lemon and vanilla)
- {b, v, v} (one of banana, two of vanilla)

(And just to be clear: There are **n=5** things to choose from, and we choose **r=3** of them.
Order does not matter, and we **can** repeat!)

Now, I can't describe directly to you how to calculate this, but I can show you a **special technique** that lets you work it out.



Think about the ice cream being in boxes, we could say "move past the first box, then take 3 scoops, then move along 3 more boxes to the end" and we will have 3 scoops of chocolate!

So it is like we are ordering a robot to get our ice cream, but it doesn't change anything, we still get what we want.

We could write this down as →○○○→→→(arrow means **move**, circle means **scoop**).

In fact the three examples above would be written like this:

{c, c, c} (3 scoops of chocolate):	→○○○→→→
{b, l, v} (one each of banana, lemon and vanilla):	○→→○→→○
{b, v, v} (one of banana, two of vanilla):	○→→→→○○

OK, so instead of worrying about different flavors, we have a ***simpler*** question: "how many different ways can we arrange arrows and circles?"

Notice that there are always 3 circles (3 scoops of ice cream) and 4 arrows (we need to move 4 times to go from the 1st to 5th container).

So (being general here) there are ***r* + *(n-1)*** positions, and we want to choose ***r*** of them to have circles.

This is like saying "we have ***r* + *(n-1)*** pool balls and want to choose ***r*** of them". In other words it is now like the pool balls question, but with slightly changed numbers. And we would write it like this:

$$\binom{n+r-1}{r} = \frac{(n+r-1)!}{r!(n-1)!}$$

where ***n*** is the number of things to choose from, and we choose ***r*** of them
(Repetition allowed, order doesn't matter)

Interestingly, we could have looked at the arrows instead of the circles, and we would have then been saying "we have ***r* + *(n-1)*** positions and want to choose ***(n-1)*** of them to have arrows", and the answer would be the same ...

$$\binom{n+r-1}{r} = \binom{n+r-1}{n-1} = \frac{(n+r-1)!}{r!(n-1)!}$$

So, what about our example, what is the answer?

$$\frac{\mathbf{(5+3-1)!}}{\mathbf{3!(5-1)!}} = \frac{\mathbf{7!}}{\mathbf{3! \times 4!}} = \frac{\mathbf{5040}}{\mathbf{6 \times 24}} = \mathbf{35}$$

In Conclusion

Phew, that was a lot to absorb, so maybe you could read it again to be sure!

But knowing *how* these formulas work is only half the battle. Figuring out how to interpret a real world situation can be quite hard.

But at least now you know how to calculate all 4 variations of "Order does/does not matter" and "Repeats are/are not allowed".