Lecture 32

Adomian Decomposition Method

The Adomian decomposition method was introduced and developed by George Adomian and is well addressed in the literature. The Adomian decomposition method has been receiving much attention in recent years in applied mathematics in general, and in the area of series solutions in particular. The method proved to be powerful, effective, and can easily handle a wide class of linear or nonlinear, ordinary or partial differential equations, and linear and nonlinear integral equations. The decomposition method demonstrates fast convergence of the solution and therefore provides several significant advantages.

The Adomian decomposition method consists of decomposing the unknown function *u(x, y)* of any equation into a sum of an infinite number of components defined by the decomposition series

$$u(x,y)=\sum_{n=0}^{\infty}u_n(x,y),$$

where the components $u_n(x,y), n\geq 0$ are to be determined in a recursive manner. The decomposition method concerns itself with finding the components $u_0, u_1, u_2, \ldots$ individually.

To have a clear overview of Adomian decomposition method, first consider

$$Fu = g(t),$$

where *F* is a nonlinear ordinary differential operator with linear and nonlinear terms. We could represent the linear term by *Lu*+*Ru* where *L* is the linear operator. We choose *L* as the highest ordered derivative, which is assumed to be invertible. The remainder of the linear operator is *R*. The nonlinear term is represented by *f* (*u*). Thus

$$Lu+Ru+f(u)=g \qquad (1)$$

$$Lu=g-Ru-f(u) \qquad (2)$$

After applying the inverse operator L^{-1} to both sides of above equation, we have

$$L^{-1}Lu=u=L^{-1}g-L^{-1}Ru-L^{-1}f(u) \qquad (3)$$

The decomposition method consists in looking for the solution in the series form $u=\sum_{n=0}^{\infty}u_n$. The nonlinear operator is decomposed as

$$f(u)=\sum_{n=0}^{\infty}A_n,$$

Where A_n depends on $u_0, u_1, u_2, \ldots, u_n$, called the Adomian polynomials that are obtained by writing

$$u(\lambda) = \sum_{n=0}^{\infty} u_n \lambda^n; \quad f(u(\lambda)) = \sum_{n=0}^{\infty} A_n \lambda^n, \tag{4}$$

Where λ is a parameter. From eq.(4), A_n*'s* are deduced as

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[f\left(\sum_{n=0}^{\infty} u_n \lambda^n \right) \right]_{\lambda=0} \tag{5}$$

The first few Adomian polynomials are

$$\begin{aligned}
A_0 &= f(u_0), \\
A_1 &= u_1 f'(u_0), \\
A_2 &= u_2 f'(u_0) + \frac{u_1^2}{2!} f''(u_0), \\
A_3 &= u_3 f'(u_0) + u_1 u_2 f''(u_0) + \frac{u_1^3}{3!} f^{(3)}(u_0), \\
&\ldots
\end{aligned} \tag{6}$$

By substituting eq.(4) into (3), The decomposition method consists in identifying the u_n's by means of the formulae

$$\sum_{n=0}^{\infty} u_n = L^{-1} g - L^{-1} R \sum_{n=0}^{\infty} u_n - L^{-1} \left(\sum_{n=0}^{\infty} A_n \right)$$

Through using Adomian decomposition method, the components $u_n(x)$ can be determined as

$$\begin{cases} u_0 = L^{-1} g \\ u_{n+1} = -L^{-1} R u_n - L^{-1} A_n \end{cases} \tag{7}$$

Hence, the series solution of *u(x)* can be obtained by using above equations.

For numerical purposes, the n-term approximant

$$\psi_n = \sum_{n=0}^{n-1} u_k$$

can be used to approximate the exact solution.

A REFERENCE LIST OF THE ADOMIAN POLYNOMIALS

$$A_0 = f(u_0),$$

$$A_1 = u_1 f'(u_0),$$

$$A_2 = u_2 f'(u_0) + \frac{u_1^2}{2!} f''(u_0),$$

$$A_3 = u_3 f'(u_0) + u_1 u_2 f''(u_0) + \frac{u_1^3}{3!} f^{(3)}(u_0),$$

$$A_4 = u_4 f'(u_0) + \left[\frac{u_1^2}{2!} + u_1 u_3\right] f''(u_0) + \frac{u_1^2 u_2}{2!} f^{(3)}(\mathrm{u}_0) + \frac{u_1^4}{4!} f^{(4)}(u_0),$$

$$A_5 = u_5 f'(\mathrm{u}_0) + \left[u_2 u_3 + u_1 u_4\right] f''(\mathrm{u}_0) + \left[\frac{u_1 u_2^2}{2!} + \frac{u_1^2 u_3}{2!}\right] f^{(3)}(\mathrm{u}_0)$$

$$+ \frac{u_1^3 u_2}{3!} f^{(4)}(\mathrm{u}_0) + \frac{u_1^5}{5!} f^{(5)}(\mathrm{u}_0),$$

$$A_6 = u_6 f'(\mathrm{u}_0) + \left[\frac{u_3^2}{2!} + u_2 u_4 + u_1 u_5\right] f''(\mathrm{u}_0) + \left[\frac{u_2^3}{3!} + u_1 u_2 u_3 + \frac{u_1^2 u_4}{2!}\right] f^{(3)}(\mathrm{u}_0)$$

$$+ \left[\frac{u_1^2}{2!}\frac{u_2^2}{2!} + \frac{u_1^3 u_3}{3!}\right] f^{(4)}(\mathrm{u}_0) + \frac{u_1^4 u_2}{4!} f^{(5)}(\mathrm{u}_0) + \frac{u_1^6}{6!} f^{(6)}(\mathrm{u}_0),$$

$$A_7 = u_7 f'(\mathrm{u}_0) + \left[u_3 u_4 + u_2 u_5 + u_1 u_6\right] f''(\mathrm{u}_0) + \left[\frac{u_2^2 u_3}{2!} + \frac{u_1 u_3^2}{2!} + u_1 u_2 u_4 + \frac{u_1^2 u_5}{2!}\right] f^{(3)}(\mathrm{u}_0)$$

$$+ \left[\frac{u_1 u_2^3}{3!} + \frac{u_1^2 u_2 u_3}{2!} + \frac{u_1^3 u_4}{3!}\right] f^{(4)}(\mathrm{u}_0) + \left[\frac{u_1^3}{3!}\frac{u_2^2}{2!} + \frac{u_1^4 u_3}{4!}\right] f^{(5)}(\mathrm{u}_0)$$

$$+ \frac{u_1^5 u_2}{5!} f^{(6)}(\mathrm{u}_0) + \frac{u_1^7}{7!} f^{(7)}(\mathrm{u}_0),$$

$$A_8 = u_8 f'(\mathrm{u}_0) + \left[\frac{u_4^2}{2!} + u_3 u_5 + u_2 u_6 + u_1 u_7\right] f''(\mathrm{u}_0)$$

$$+ \left[\frac{u_2 u_3^2}{2!} + \frac{u_2^2 u_4}{2!} + u_1 u_3 u_4 + u_1 u_2 u_5 + \frac{u_1^2 u_6}{2!}\right] f^{(3)}(\mathrm{u}_0)$$

$$+ \left[\frac{u_2^4}{4!} + \frac{u_1 u_2^2 u_3}{2!} + \frac{u_1^2}{2!}\frac{u_3^2}{2!} + \frac{u_1^2 u_2 u_4}{2!} + \frac{u_1^3 u_5}{3!}\right] f^{(4)}(\mathrm{u}_0)$$

$$+ \left[\frac{u_1^2}{2!}\frac{u_2^3}{3!} + \frac{u_1^3 u_2 u_3}{3!} + \frac{u_1^4 u_4}{4!}\right] f^{(5)}(\mathrm{u}_0) + \left[\frac{u_1^4}{4!}\frac{u_2^2}{2!} + \frac{u_1^5 u_3}{5!}\right] f^{(6)}(\mathrm{u}_0) + \frac{u_1^6 u_2}{6!} f^{(7)}(\mathrm{u}_0) + \frac{u_1^8}{8!} f^{(8)}(\mathrm{u}_0),$$

$$A_9 = u_9 f'(\mathrm{u}_0) + \left[u_4u_5 + u_3u_6 + u_2u_7 + u_1u_8\right] f''(\mathrm{u}_0)$$
$$+\left[\frac{u_3^3}{3!} + u_2u_3u_4 + \frac{u_2^2u_5}{2!} + \frac{u_1u_4^2}{2!} + u_1u_3u_5 + u_1u_2u_6 + \frac{u_1^2u_7}{2!}\right] f^{(3)}(\mathrm{u}_0)$$
$$+\left[\frac{u_2^3u_3}{3!} + \frac{u_1u_2u_3^2}{2!} + \frac{u_1u_2^2u_4}{2!} + \frac{u_1^2u_3u_4}{2!} + \frac{u_1^2u_2u_5}{2!} + \frac{u_1^3u_6}{3!}\right] f^{(4)}(\mathrm{u}_0)$$
$$+\left[\frac{u_1u_2^4}{4!} + \frac{u_1^2}{2!}\frac{u_2^2u_3}{2!} + \frac{u_1^3}{3!}\frac{u_3^2}{2!} + \frac{u_1^3u_2u_4}{3!} + \frac{u_1^4u_5}{4!}\right] f^{(5)}(\mathrm{u}_0)$$
$$+\left[\frac{u_1^3}{3!}\frac{u_2^3}{3!} + \frac{u_1^4u_2u_3}{4!} + \frac{u_1^5u_4}{5!}\right] f^{(6)}(\mathrm{u}_0) + \left[\frac{u_1^5}{5!}\frac{u_2^2}{2!} + \frac{u_1^6u_3}{6!}\right] f^{(7)}(\mathrm{u}_0)$$
$$+\frac{u_1^7u_2}{7!} f^{(8)}(\mathrm{u}_0) + \frac{u_1^9}{9!} f^{(9)}(\mathrm{u}_0)$$

$$A_{10} = u_{10} f'(\mathrm{u}_0) + \left[\frac{u_5^2}{2!} + u_4u_6 + u_3u_7 + u_2u_8 + u_1u_9\right] f''(\mathrm{u}_0)$$
$$+\left[\frac{u_3^2u_4}{2!} + \frac{u_4^2u_2}{2!} + u_2u_3u_5 + \frac{u_2^2u_6}{2!} + u_1u_4u_5 + u_1u_3u_6 + u_1u_2u_7 + \frac{u_1^2u_8}{2!}\right] f^{(3)}(\mathrm{u}_0)$$
$$+\left[\frac{u_2^2}{2!}\frac{u_3^2}{2!} + \frac{u_2^3u_4}{3!} + \frac{u_1u_3^3}{3!} + u_1u_2u_3u_4 + \frac{u_1u_2^2u_5}{2!} + \frac{u_1^2}{2!}\frac{u_4^2}{2!} + \frac{u_1^2u_3u_5}{2!} + \frac{u_1^2u_2u_6}{2!} + \frac{u_1^3u_7}{3!}\right] f^{(4)}(\mathrm{u}_0)$$
$$+\left[\frac{u_2^5}{5!} + \frac{u_1u_2^3u_3}{3!} + \frac{u_1^2u_2}{2!}\frac{u_3^2}{2!} + \frac{u_1^2}{2!}\frac{u_2^2u_4}{2!} + \frac{u_1^3u_3u_4}{3!} + \frac{u_1^3u_2u_5}{3!} + \frac{u_1^4u_6}{4!}\right] f^{(5)}(\mathrm{u}_0)$$
$$+\left[\frac{u_1^2}{2!}\frac{u_2^4}{4!} + \frac{u_1^3}{3!}\frac{u_2^2u_3}{2!} + \frac{u_1^4}{4!}\frac{u_3^2}{2!} + \frac{u_1^4u_2u_4}{4!} + \frac{u_1^5u_5}{5!}\right] f^{(6)}(\mathrm{u}_0)$$
$$+\left[\frac{u_1^4}{4!}\frac{u_2^3}{3!} + \frac{u_1^5u_2u_3}{5!} + \frac{u_1^6u_4}{6!}\right] f^{(7)}(\mathrm{u}_0) + \left[\frac{u_1^6}{6!}\frac{u_2^2}{2!} + \frac{u_1^7u_3}{7!}\right] f^{(8)}(\mathrm{u}_0)$$
$$+\frac{u_1^8u_2}{8!} f^{(9)}(\mathrm{u}_0) + \frac{u_1^{10}}{10!} f^{(10)}(\mathrm{u}_0)$$
$$\vdots$$

Examples: Calculate the Adomian polynomials for following non-linear functions

1. $f(u) = u^5$

Solution

The Adomian polynomials are determined by using the above reference list of formulas, as

$$
\begin{aligned}
A_0 &= u_0^5 \\
A_1 &= 5u_0^4 u_1 \\
A_2 &= 5u_0^4 u_2 + 10u_0^3 u_1^2 \\
A_3 &= 5u_0^4 u_3 + 20u_0^3 u_1 u_2 + 10u_0^2 u_1^3 \\
A_4 &= 5u_0^4 u_4 + 5u_1^4 u_0 + 10u_0^3 u_2^2 + 20u_0^3 u_1 u_3 + 30u_0^2 u_1^2 u_2 \\
A_5 &= u_1^5 + 5u_0^4 u_5 + 20u_0^3 u_1 u_4 + 20u_0^3 u_2 u_3 + 20u_1^3 u_0 u_2 + 30u_0^2 u_2^2 u_1 + 30u_0^2 u_1^2 u_3 \\
A_6 &= 5u_0^4 u_6 + 5u_1^4 u_2 + 10u_0^3 u_3^2 + 10u_0^2 u_2^3 + 20u_0^3 u_1 u_5 + 20u_0^3 u_2 u_4 + 20u_1^3 u_0 u_3 \\
&\quad + 30u_0^2 u_1^2 u_4 + 30u_1^2 u_2^2 u_0 + 60u_0^2 u_1 u_2 u_3 \\
A_7 &= 5u_0^4 u_7 + 5u_1^4 u_3 + 10u_1^3 u_2^2 + 20u_0^3 u_1 u_6 + 20u_0^3 u_2 u_5 + 20u_0^3 u_3 u_4 + 20u_2^3 u_1 u_0 \\
&\quad + 20u_1^3 u_0 u_4 + 30u_0^2 u_2^2 u_3 + 30u_0^2 u_3^2 u_1 + 30u_0^2 u_1^2 u_5 + 60u_0^2 u_1 u_2 u_4 + 60u_1^2 u_0 u_2 u_3
\end{aligned}
$$

Remark

Notice that for u^m each individual term is the product of m factors. Each term of A_n has five factors--- the sum of superscripts is m (or 5 in this case). The sum of subscripts is n. The second term of A_4, as an example, is $5u_1u_1u_1u_1u_0$ and the sum of subscripts is 4. A very convenient check on the numerical coefficients in each term is the following. Each coefficient is $m!$ divided by the product of factorials of the superscripts for a given term. Thus, the second term of $A_3(u^5)$ has the coefficient $5!(2!)(2!)(1!) = 30$.

2. $f(u) = u^2$

Solution

$$
\begin{aligned}
A_0 &= u_0^2 \\
A_1 &= 2u_0u_1 \\
A_2 &= u_1^2 + 2u_0u_2 \\
A_3 &= 2u_1u_2 + 2u_0u_3 \\
A_4 &= u_2^2 + 2u_1u_3 + 2u_0u_4 \\
A_5 &= 2u_2u_3 + 2u_1u_4 + 2u_0u_5
\end{aligned}
$$

3. $f(\theta) = \sin\theta$

Solution

$$
\begin{aligned}
A_0 &= \sin\theta_0 \\
A_1 &= \theta_1 \cos\theta_0 \\
A_2 &= -\left(\theta_1^2 / 2\right)\sin\theta_0 + \theta_2 \cos\theta_0 \\
A_3 &= -\left(\theta_1^3 / 6\right)\cos\theta_0 - \theta_1\theta_2 \sin\theta_0 + \theta_3 \cos\theta_0 \\
&\;\;\vdots
\end{aligned}
$$

Remark:

The essential features of the decomposition method for linear and nonlinear, homogeneous and inhomogeneous equations, can be outlined as follows:

- Express the partial differential equation, linear or nonlinear, in an operator form.
- Apply the inverse operator to both sides of the equation written in an operator form.
- Set the unknown function $u(x, y)$ into a decomposition series
  $$u = \sum_{n=0}^{\infty} u_n$$
  whose components are elegantly determined. We next substitute the above series into both sides of the resulting equation.
- Identify the zeroth component $u_0(x, y)$ as the terms arising from the given conditions and from integrating the source term $g(x, y)$, both are assumed to be known.

- Determine the successive components of the series solution u_k, $k \geq 1$ by applying the recursive scheme (6), where each component u_k can be completely determined by using the previous component u_{k-1}.
- Substitute the determined components into $u = \sum_{n=0}^{\infty} u_n$ to obtain the solution in a series form. An exact solution can be easily obtained in many equations if such a closed form solution exists.

It is to be noted that Adomian decomposition method approaches any equation, homogeneous or inhomogeneous, and linear or nonlinear in a straightforward manner without any need to restrictive assumptions such as linearization, discretization or perturbation. There is no need in using this method to convert inhomogeneous conditions to homogeneous conditions as required by other techniques.

Example 4:

Solve the following homogeneous differential equation by using Adomian decomposition method.

$$u'(x) = u(x), \;\; u(0) = \mathrm{A}. \tag{8}$$

Solution

In an operator form the given equation becomes

$$Lu = u, \tag{9}$$

where L is the differential operator given by

$$L = \frac{d}{dx}, \tag{10}$$

and therefore the inverse operator L^{-1} is defined by

$$L^{-1}(.) = \int_0^x (.)dx. \tag{11}$$

Applying L^{-1} to both sides of (9) and using the initial condition, we obtain

$$L^{-1}(Lu) = L^{-1}(u)$$

$$\Rightarrow u(x) - u(0) = L^{-1}(u)$$

$$\Rightarrow u(x) = A + L^{-1}(u) \tag{12}$$

Substituting the series assumption, $u = \sum_{n=0}^{\infty} u_n$ into both sides of above equation

$$\sum_{n=0}^{\infty} u_n(x) = A + L^{-1}\left(\sum_{n=0}^{\infty} u_n(x)\right) \qquad (13)$$

In view of (13), we have the following recursive relation

$$\begin{aligned} u_\circ(x) &= A, \\ u_{k+1}(x) &= L^{-1}(u_k(x)); \quad k \geq 0. \end{aligned} \qquad (14)$$

Consequently, we obtain

$$\begin{aligned} u_\circ(\mathrm{x}) &= \mathrm{A}, \\ u_1(\mathrm{x}) &= \mathrm{L}^{-1}(\mathrm{u}_\circ(\mathrm{x})) = Ax, \\ u_2(\mathrm{x}) &= \mathrm{L}^{-1}(\mathrm{u}_1(\mathrm{x})) = \frac{Ax^2}{2}, \\ u_3(\mathrm{x}) &= \mathrm{L}^{-1}(\mathrm{u}_2(\mathrm{x})) = \frac{Ax^3}{6}, \\ &\vdots \end{aligned} \qquad (15)$$

Hence $u = \sum_{n=0}^{\infty} u_n$ gives the solution in a series form as

$$\begin{aligned} u(x) &= A\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) \\ u(x) &= Ae^x. \end{aligned}$$

Example 5:

Solve the following homogeneous differential equation by using Adomian decomposition method.

$$u''(x) = xu; \quad u(0) = A, \;\; u'(0) = B. \qquad (16)$$

Solution

In an operator form the given equation becomes

$$Lu = xu, \qquad (17)$$

where L is the differential operator given by

$$L(.) = \frac{d^2}{dx^2}(.), \tag{18}$$

and therefore, the inverse operator L^{-1} is defined by

$$L^{-1}(.) = \int_0^x \int_0^x (.)dxdx. \tag{19}$$

Applying L^{-1} to both sides of (17) and using the initial conditions, we obtain

$$L^{-1}(Lu) = L^{-1}(xu)$$

So that

$$u(x) - xu'(0) - u(0) = L^{-1}(xu)$$
$$\Rightarrow u(x) = A + Bx + L^{-1}(xu) \tag{20}$$

Substituting the series assumption, $u = \sum_{n=0}^{\infty} u_n$ into both sides of above equation

$$\sum_{n=0}^{\infty} u_n(x) = A + Bx + L^{-1}\left(x \sum_{n=0}^{\infty} u_n(x) \right) \tag{21}$$

Following the decomposition method we obtain the following recursive relation

$$\begin{aligned} u_\circ(x) &= A + Bx, \\ u_{k+1}(x) &= L^{-1}(xu_k(x)); \quad k \geq 0. \end{aligned} \tag{22}$$

Consequently, we obtain

$$\begin{aligned} u_\circ(x) &= A + Bx, \\ u_1(x) &= L^{-1}(xu_\circ) = \frac{Ax^3}{6} + \frac{Bx^4}{12}, \\ u_2(x) &= L^{-1}(xu_1) = \frac{Ax^6}{180} + \frac{Bx^7}{504}, \\ &\vdots \end{aligned} \tag{23}$$

Thus, the solution in series form is

$$u(x) = A\left(1 + \frac{x^3}{6} + \frac{x^6}{180} + \cdots\right) + B\left(x + \frac{x^4}{12} + \frac{x^7}{504} + \cdots\right).$$

Example 6:

Solve the following homogeneous partial differential equation by using Adomian decomposition method.

$$u_x + u_y = x + y, \tag{24}$$
$$u(x,0) = 0,\ u(0,y) = 0.$$

Solution

In an operator form PDE becomes

$$L_x u(x,y) = x + y - L_y u(x,y), \tag{25}$$

where the operators are defined as

$$L_x = \frac{\partial}{\partial x},\ L_y = \frac{\partial}{\partial y},$$

and their inverse operators are as

$$L_x^{-1}(.) = \int_0^x (.)dx, \qquad L_y^{-1}(.) = \int_0^y (.)dy.$$

The *x*-solution:

This solution can be obtained by applying L_x^{-1} to both sides of (25),

$$L_x^{-1}L_x u(x,y) = L_x^{-1}(x+y) - L_x^{-1}L_y u(x,y),$$
$$\Rightarrow u(x,y) = u(0,y) + \frac{1}{2}x^2 + xy - L_x^{-1}L_y u,$$
$$\Rightarrow u(x,y) = \frac{1}{2}x^2 + xy - L_x^{-1}L_y u. \tag{26}$$

Above Eq. is obtained on using the given condition $u(0, y) = 0$, by integrating $f(x,y) = x + y$ with respect to x and using $L_x^{-1}L_x u(x,y) = u(x,y) - u(0,y)$.

Substituting the unknown function $u(x, y)$ as an infinite number of components $u_n(x, y)$, $n \geq 0$ given by $u(x,y) = \sum_{n=0}^{\infty} u_n(x,y)$ in (26),

$$\sum_{n=0}^{\infty} u_n(x,y) = \frac{1}{2}x^2 + xy - L_x^{-1}L_y\left[\sum_{n=0}^{\infty} u_n(x,y)\right], \quad (27)$$

$$u_0 + u_1 + u_2 + u_3 + \ldots = \frac{1}{2}x^2 + xy - L_x^{-1}(L_y(u_0 + u_1 + u_2 + \ldots)).$$

Consequently, the recursive scheme that will enable us to completely determine the successive components is thus constructed by

$$\begin{aligned} u_\circ(x,y) &= \frac{1}{2}x^2 + xy, \\ u_{k+1}(x,y) &= -L_x^{-1}(L_y(u_k)); \quad k \geq 0. \end{aligned} \quad (28)$$

Which gives

$$\begin{aligned} u_\circ(x,y) &= \frac{1}{2}x^2 + xy, \\ u_1(x,y) &= -L_x^{-1}(L_y(u_0)) = -L_x^{-1}\left(L_y\left(\frac{1}{2}x^2 + xy\right)\right) = -\frac{1}{2}x^2, \\ u_2(x,y) &= -L_x^{-1}(L_y(u_1)) = -L_x^{-1}\left(L_y\left(-\frac{1}{2}x^2\right)\right) = 0. \end{aligned} \quad (29)$$

Accordingly, $u_k(x,y) = 0$, $k \geq 2$. Having determined the components of $u(x,y)$, we find

$$u = u_0 + u_1 + u_2 + u_3 + \ldots = \frac{1}{2}x^2 + xy - \frac{1}{2}x^2 = xy \quad (30)$$

the exact solution of the equation.

The y-solution:

It is important to note that the exact solution can also be obtained by finding the y-solution. In an operator form, we can write the given equation as

$$L_y u = x + y - L_x u. \quad (31)$$

By applying L_y^{-1} to both sides of (31),

$$\begin{aligned} L_y^{-1}L_y u(x,y) &= L_y^{-1}(x+y) - L_y^{-1}L_x u(x,y), \\ \Rightarrow u(x,y) &= u(x,0) + xy + \frac{1}{2}y^2 - L_y^{-1}L_x u, \\ \Rightarrow u(x,y) &= xy + \frac{1}{2}y^2 - L_y^{-1}L_x u, \end{aligned} \quad (32)$$

Using $u(x,y)=\sum_{n=0}^{\infty}u_n(x,y)$ on both sides of (32),

$$\sum_{n=0}^{\infty}u_n(x,y)=xy+\frac{1}{2}y^2-L_y^{-1}L_x\left[\sum_{n=0}^{\infty}u_n(x,y)\right], \tag{33}$$

$$u_0+u_1+u_2+u_3+\ldots=xy+\frac{1}{2}y^2-L_y^{-1}(L_x(u_0+u_1+u_2+\ldots)).$$

The recursive scheme will be defined as

$$u_\circ(x,y)=xy+\frac{1}{2}y^2,$$

$$u_{k+1}(x,y)=-L_y^{-1}(L_x(u_k));\quad k\geq 0. \tag{34}$$

So we have

$$u_\circ(x,y)=xy+\frac{1}{2}y^2,$$

$$u_1(x,y)=-L_y^{-1}(L_x(u_0))=-L_y^{-1}\left(L_x\left(xy+\frac{1}{2}y^2\right)\right)=-\frac{1}{2}y^2,$$

$$u_2(x,y)=-L_y^{-1}(L_x(u_1))=-L_y^{-1}\left(L_x\left(-\frac{1}{2}y^2\right)\right)=0. \tag{35}$$

Consequently, $u_k(x,y)=0$, $k\geq 2$. Having determined the components of $u(x,y)$, we find

$$u=u_0+u_1+u_2+u_3+\ldots=xy+\frac{1}{2}y^2-\frac{1}{2}y^2=xy \tag{36}$$

the exact solution of the equation.

Note:

The exact solution of the given PDE can be obtained by determining the *x*-solution or y-solution only as discussed above, depending upon the given equation.

Exercises

1. Calculate the Adomian polynomials for the following non-linear functions

   a. $f(u) = u^3$,

   b. $f(x) = \sinh\left(\dfrac{x}{2}\right)$,

   c. $f(u) = u^{-m}; m > 0.$

2. Use the Adomian decomposition method to show that the exact solution of following B.V.Ps can be obtained by determining the *x*-solution or the *y*-solution:

   a. $u_x - u_y = 0;\ \ u(x,0) = x,\ u(0,y) = \text{y}.$

   b. $xu_x + u_y = 3u;\ \ u(x,0) = x^2,\ u(0,y) = 0.$

   c. $u_x - yu = 0;\ \ u(0,y) = 1.$

3. Solve the following homogeneous partial differential equation by using Adomian decomposition method

$$u_x + u_y + u_z = u;$$

$$u(0,y,z) = 1 + e^y + e^z,\ u(x,0,z) = 1 + e^x + e^z,\ u(x,y,0) = 1 + e^x + e^y,$$

where $u = u(x,y,z)$.