Lecture Handouts

on

VECTORS AND CLASSICAL MECHANICS (MTH-622)

Virtual University of Pakistan
Pakistan

About the Handouts

The following books have been mainly followed to prepare the slides and handouts:

1. Spiegel, M.R., *Theory and Problems of Vector Analysis: And an Introduction to Tensor Analysis*. 1959: McGraw-Hill.
2. Spiegel, M.S., *Theory and problems of theoretical mechanics*. 1967: Schaum.
3. Taylor, J.R., *Classical Mechanics*. 2005: University Science Books.
4. DiBenedetto, E., *Classical Mechanics: Theory and Mathematical Modeling*. 2010: Birkhäuser Boston.
5. Fowles, G.R. and G.L. Cassiday, *Analytical Mechanics*. 2005: Thomson Brooks/Cole.

The first two books were considered as main text books. Therefore the students are advised to read the first two books in addition to these handouts. In addition to the above mentioned books, some other reference book and material was used to get these handouts prepared.

Contents

Module No. 79 1

Introduction to Classical Mechanics 1

Module No. 80 2

Introductions to the Basics of Newton's Law 2

Module No. 81 4

Introductions to Rectangular Components of Velocity & Acceleration 4

Module No. 82 6

Introduction to Tangential and Normal Components of Velocity & Acceleration 6

Module No. 83 8

Example of Tangential and Normal Acceleration 8

Module No. 84 10

Curvature and Radius of Curvature 10

Module No. 85 12

Introduction to Radial and Transverse Components of Velocity & Acceleration 12

Module No. 86 14

Example of Radial and Transverse Components of Velocity and Acceleration 14

Module No. 87 15

Reference System or Inertial Frame 15

Module No. 88 16

Example of inertial Frame 16

Module No. 89 18

Newton's First and Second Laws 18

Module No. 90 20

The Third Law and Law of Conservation of Momentum 20

Module No. 91 23

Validity of Newton's Law 23

Module No. 92 24

Example of Newton's Law 1, 2 24

Module No. 93 27

Example of Newton's Law: 3 27

Module No. 94 29

Introduction to Energy: Kinetic Energy 29

Module No. 95 30
Introduction to Work – Theorem 30
Module No. 96 33
Example of Work Done 33
Module No. 97 34
Conservative Force Field 34
Module No. 98 36
Example of Conservative Field 36
Module No. 99 37
Related Topic of Conservative Force Fields 37
Module No. 100 39
Non-Conservative Force Field 39

Module No. 79

Introduction to Classical Mechanics

Mechanics is the science of motion. It studies the states of rest and motion and the laws governing rest, equilibrium and motion.
A broad division of mechanics is made through the terms classical mechanics and quantum mechanics.

Classical Mechanics

The science of classical mechanics deals with the motion of objects through absolute space and time in the Newtonian sense. Classical mechanics deals with the macroscopic object i.e. those bodies which we encounter in everyday life. Such bodies may be those bodies in the form of billiard balls, parts of machinery and astronomical bodies.

Classical mechanics itself has two broad divisions in the form of non-relativistic and relativistic mechanics. Non-relativistic mechanics, based on the laws of Newton is concerned with bodies moving at speeds and velocity negligibly small as compared to the speed of light $c = 3 \times 10^5$per kilometer per second. Relativistic mechanics is dispensable for particles and bodies moving with speed comparable to the speed of light.

Quantum mechanics

The quantum mechanics is the mechanics of microscopic object. Its basic concepts, principals and laws are of entirely different from those of classical mechanics.

Division of Classical Mechanics

Three major divisions of classical mechanics are the following:

Mechanics of particles and rigid bodies: It is based on newton's law. Basic concepts and terms are space, time and mass; particle and body; velocity, momentum and acceleration; force and energy.

Mechanics of fluid: it is also based on newton's law and their extensions and deal with the behavior of the fluid (liquid and gases) in motion. Its two well-known branches are hydrodynamics (for fluid) and Aerodynamics (for gases).

Mechanics of elastic solids: it deals with the behavior of solids when the undergo deformation under forces.

Module No. 80

Introductions to the Basics of Newton's Law

Newton's three law of motion are formulated in terms of crucial underlying concepts: the notations of space, time, mass and force.
In this section we will review all of the factors of formulation of newton's law.

Space and time:

We shall assume that space and time are described strictly in the Newtonian sense.

- **Space**. This is closely related to the concepts of point, position,' direction and displacement sense. Three- dimensional space is Euclidian, and positions of points in that space are specified by a set of three numbers (x,y,z) relative to the origin (0,0,0) of a rectangular Cartesian coordinate system. Measurement in space involves the concepts of length or distance, with which we assume familiarity. Units of length are feet, meters, miles, etc.
- **Time**. This concept is derived from our experience of having one event taking place after, before or simultaneous with another event. Measurement of time is achieved, for example, by use of clocks. Units of time are seconds, hours, years, etc. the basic unit of time in SI, began as an arbitrary fraction (1/86,400) of a mean solar day ($24 \chi 60 \text{ x } 60 = 86{,}400$).
- **Mass** Physical objects are composed of "small bits of matter" such as atoms and molecules. From this we arrive at the concept of a material object called a particle which can be considered as occupying a point in space and perhaps moving as time goes by. A measure of the "quantity of matter" associated with a particle is called its mass. The mass of an object is characterized by its inertia. It's the resistance to be accelerated. Units of mass are grams, kilograms, etc. Unless otherwise stated we shall assume that the mass of a particle does not change with time. Length, mass and time are often called dimensions from which other physical quantities are constructed.

- **Force** The informal notion of force as a push or pull. It is a vector quantity. As the unit of force we naturally adopt the newton (abbreviated N) defined as the magnitude of any single force that accelerates a standard kilogram mass with an acceleration of 1 m/s 2. If we apply a given force F (and no other forces) to any object at rest, the direction of F is defined as the direction of the resulting acceleration, that is, the direction in which the body moves off.

Now that we know, at least in principle, what we mean by positions, times, masses, and forces, we can proceed to discuss the cornerstone of our subject — Newton's three laws of motion.

Module No. 81

Introductions to Rectangular Components of Velocity & Acceleration

Rectangular Components

The process of splitting a vector into various parts or components is called "Resolution of vector" and these parts are called components of vector. If we split a vector in a rectangular plane OXY, such components are called rectangular components of a vector.

Component Along x-axis is called horizontal component of vector.

Component Along y-axis is called vertical component of vector.

Position vector

It is often convenient to describe the motion of a particle in terms of its x, y or rectangular components, relative to a fixed frame of reference. In a given reference system, the position of a particle can be specified by a single vector, namely, the displacement of the particle relative to the origin of the coordinate system. This vector is called the position vector of the particle. In rectangular coordinates (see Figure), the position vector is simply

$$r = x\boldsymbol{i} + y\boldsymbol{j}$$

The components of the position vector of a moving particle are functions of the time, namely,

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{t})\,,\boldsymbol{y} \;=\; \boldsymbol{y}(\boldsymbol{t})$$

RECTANGULAR COMPONENTS: VELOCITY

In above Equation we gave the formal definition of the derivative of any vector with respect to some parameter. In particular, if the vector is the position vector r of a moving particle and the parameter is the time t, the derivative of r with respect to t is called the velocity, which we shall denote by

$$\boldsymbol{v} = \frac{\boldsymbol{dr}}{\boldsymbol{dt}} = \boldsymbol{i\dot{x}} + \boldsymbol{j\dot{y}} + \boldsymbol{k\dot{z}}$$

where the dots indicate differentiation with respect to t.

where

$$v_x = \dot{x}, v_y = \dot{y} \quad , v_z = \dot{y}$$

The magnitude of the velocity is called the speed. In rectangular components the speed is just

The magnitude of the velocity vector is

$$v = (v_x{}^2 + v_y{}^2 + v_z{}^2)^{1/2}$$

The direction of ***v*** is tangent to the path of motion.

RECTANGULAR COMPONENTS: ACCELERATION

Acceleration represents the rate of change in the velocity of a particle.

If a particle's velocity changes from v to v' over a time increment Δt, the average acceleration during that increment is: avg ***a*** = Δv/Δt = (v-v')/Δt the instantaneous accelerations the time derivative of velocity:

$$a = \frac{dv}{dt} = \frac{d^2r}{dt^2} = ia_x + ja_y + ka_z$$

In rectangular components,

$$a = \dot{v_x}\imath + \dot{v_y}\jmath + \dot{v_z}k$$

$$a = i\ddot{x} + j\ddot{y} + k\ddot{z}$$

$$a_x = \ddot{x}, a_y = \ddot{y}, a_z = \ddot{z}$$

Thus, acceleration is a vector quantity whose components, in rectangular coordinates, are the second derivatives of the positional coordinates of a moving particle.

The direction of ***a*** *is* usually not tangent to the path of the particle.

Module No. 82

Introduction to Tangential and Normal Components of Velocity & Acceleration

Introduction To Tangential And Normal Components Of Vector

In mathematics, given a vector at a point on a curve, that vector can be decomposed uniquely as a sum of two vectors, one tangent to the curve, called the tangential component of the vector, and another one perpendicular to the curve, called the normal component of the vector.

Introduction to tangential and normal components of velocity

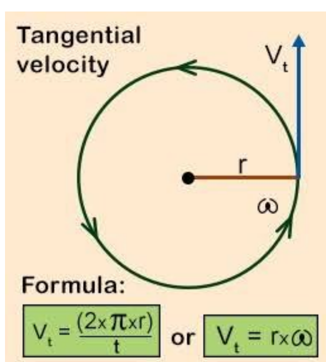
The tangential velocity is the velocity measured at any point tangent to a circular path. Thus tangential velocity, Vt is related to the angular velocity of the wheel, ω, and the radius of the wheel, r.

$$Vt = \omega r$$

Vt = tangential velocity

ω = angular velocity

r = radius of wheel



(1)

Total velocity of a particle at a point on a curve can be write as

$$v = v_t + v_N$$

We can determine normal component of velocity by

$$v_N = v - v_t$$

Tangential and normal components of Acceleration

The rate of change of tangential velocity of an object traveling in a circular orbit or path is known as Tangential acceleration. It is directed towards tangent to the path of a body.

Total acceleration of a particle at a point on a curve can be write as

$$\mathrm{a} = a_t + a_N$$

where

$a_t = \frac{dv_t}{dt}, a_N = \frac{v^2}{r}$

where v is total velocity of the object and r is radius of circular path.

Module No. 83

Example of Tangential and Normal Acceleration

Example

Consider the space curve C with the position vector $\boldsymbol{r}$ given to be at time $\boldsymbol{t}$

$$\boldsymbol{r} = 5\cos 3\boldsymbol{t}\,\boldsymbol{i} + 3\sin 3\boldsymbol{t}\,\boldsymbol{j} + 10\boldsymbol{t}\boldsymbol{k}$$

(a) Find a unit tangent vector **T** to the curve.

(b) Verify that $\boldsymbol{v}$ = v $\boldsymbol{T}$.

Solution:

(a) Tangent vector to the space curve C is

$$\frac{\boldsymbol{dr}}{\boldsymbol{dt}} = -15\sin 3\boldsymbol{t}\,\boldsymbol{i} + 9\cos 3\boldsymbol{t}\,\boldsymbol{j} + 10\boldsymbol{k}$$

The magnitude of this vector is

$$\mathrm{v} = \left|\frac{\boldsymbol{dr}}{\boldsymbol{dt}}\right| = \frac{\boldsymbol{ds}}{\boldsymbol{dt}} = \sqrt{(-15\sin 3\boldsymbol{t})^2 + (9\cos 3\boldsymbol{t})^2 + (10)^2}$$

$$= \sqrt{225\sin^2 3\boldsymbol{t} + 81\cos^2 3\boldsymbol{t} + 100}$$

The unit tangent vector to C is

$$\boldsymbol{T} = \frac{\boldsymbol{dr}/\boldsymbol{dt}}{|\boldsymbol{dr}/\boldsymbol{dt}|} = \frac{-15\sin 3\boldsymbol{t}\,\boldsymbol{i} + 9\cos 3\boldsymbol{t}\,\boldsymbol{j} + 10\boldsymbol{k}}{\sqrt{225\sin^2 3\boldsymbol{t} + 81\cos^2 3\boldsymbol{t} + 100}}$$

(b) From the first part, it follows that This follows directly from (a) since

$$\boldsymbol{v} = \frac{\boldsymbol{dr}}{\boldsymbol{dt}} = -15\sin 3\boldsymbol{t}\,\boldsymbol{i} + 9\cos 3\boldsymbol{t}\,\boldsymbol{j} + 10\boldsymbol{k}$$

$$\boldsymbol{v} = -\mathbf{15}\,\mathbf{sin}\,\mathbf{3}\boldsymbol{t}\,\boldsymbol{i} + \mathbf{9}\,\mathbf{cos}\,\mathbf{3}\boldsymbol{t}\,\boldsymbol{j} + \mathbf{10}\boldsymbol{k}\frac{\sqrt{\mathbf{225sin^2 3}\boldsymbol{t} + \mathbf{81cos^2 3}\boldsymbol{t} + \mathbf{100}}}{\sqrt{\mathbf{225sin^2 3}\boldsymbol{t} + \mathbf{81cos^2 3}\boldsymbol{t} + \mathbf{100}}}$$

$$\boldsymbol{v} = \sqrt{\mathbf{225sin^2 3}\boldsymbol{t} + \mathbf{81cos^2 3}\boldsymbol{t} + \mathbf{100}}\,\frac{(-\mathbf{15}\,\mathbf{sin}\,\mathbf{3}\boldsymbol{t}\,\boldsymbol{i} + \mathbf{9}\,\mathbf{cos}\,\mathbf{3}\boldsymbol{t}\,\boldsymbol{j} + \mathbf{10}\boldsymbol{k})}{\sqrt{\mathbf{225sin^2 3}\boldsymbol{t} + \mathbf{81cos^2 3}\boldsymbol{t} + \mathbf{100}}}$$

$$= \mathrm{v}\,\boldsymbol{T}$$

Module No. 84

Curvature and Radius of Curvature

Curvature is the amount by which a geometric object such as a surface differs from being a flat plane, or a curve from being straight as in the case of a line.

Statement:

If T is a unit tangent vector to a space curve C, then show that $\frac{dT}{ds}$ is normal to T.

Proof:

Since **T** is a unit vector, we have **T. T = 1**. Then differentiating with respect to **s**, we obtain

$$T.\frac{dT}{ds} + \frac{dT}{ds}.T = 0$$

$$2\frac{dT}{ds}.T = 0$$

$$\frac{dT}{ds}.T = 0$$

Which states that $\frac{dT}{ds}$ is normal, i.e. perpendicular to T.

If N is a unit vector in the direction of $\frac{dT}{ds}$, we have

$\frac{dT}{ds} = kN$

and we call N the unit principal normal to C. The scalar k = $\left|\frac{dT}{ds}\right|$ is called the curvature,

while $R = {}^{1}/_{k}$ is called the radius of curvature.

Example:

Find the

a. Curvature,
b. Radius of curvature
c. Unit principal normal N to any point of the space curve of given unit tangent vector is

$$T = -\frac{3}{5}\sin 2\,t\boldsymbol{i} + \frac{3}{5}\cos 2\,t\boldsymbol{j} + \frac{4}{5}\boldsymbol{k}$$

moving with 10 ms^{-1}.

Solution:

Since given tangential vector is

$$T = -\frac{3}{5}\sin 2\,t\boldsymbol{i} + \frac{3}{5}\cos 2\,t\boldsymbol{j} + \frac{4}{5}\boldsymbol{k}$$

Then,

$$\frac{dT}{ds} = \frac{\frac{dT}{dt}}{\frac{ds}{dt}} = \frac{-\left({}^{6}\!/_{5}\right)\cos 2t\; i - ({}^{6}\!/_{5})\sin 2t\; j}{10}$$

$$= -\frac{3}{25}\cos 2t\;\boldsymbol{i} - \frac{3}{25}\sin 2t\;\boldsymbol{j}$$

Thus the curvature is $k = \left|\frac{dT}{ds}\right| = \sqrt{(-\frac{3}{25}\cos 2t)^2 + (-\frac{3}{25}\sin 2t)^2} = \frac{3}{25}$

Radius of curvature $= \frac{1}{k} = \frac{25}{3}$

We have unit principal normal N = $\frac{1}{k}.\frac{dT}{ds} = \frac{25}{3}(-\frac{3}{25}\cos 2t\;\boldsymbol{i} - \frac{3}{25}\sin 2t\;\boldsymbol{j})$

$$= -\cos 2t\;\boldsymbol{i} - \sin 2t\;\boldsymbol{j}$$

Module No. 85

Introduction to Radial and Transverse Components of Velocity & Acceleration

It is often convenient to employ polar coordinates r, θ to express the position of a particle moving in a plane. Vectorially, the position of the particle can be written as the product of the radial distance r by a unit radial vector:

$$\boldsymbol{r} = r\boldsymbol{e_r}$$

As the particle moves, both r and e_r vary; thus, they are both functions of the time.
Hence, if we differentiate with respect to t, we have

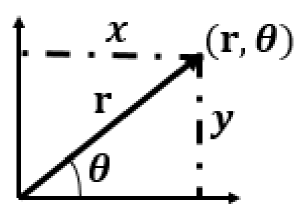
$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{r}\mathbf{e}_r + r\frac{d\mathbf{e}_r}{dt}$$

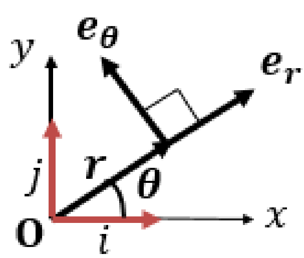
$$\mathbf{v} = \dot{r}\boldsymbol{e_r} + r\dot{\boldsymbol{e}}_{\boldsymbol{r}}$$

From fig $\boldsymbol{e_r} = \cos\theta\, i + \sin\theta\, j$ *and* $\boldsymbol{e_\theta} = -\sin\theta\, i + \cos\theta\, j$

So, $\dot{\boldsymbol{e}}_{\boldsymbol{r}} = -\sin\theta\,\dot{\theta} i + \cos\theta\dot{\theta}\, j$ or $\dot{\boldsymbol{e}}_{\boldsymbol{r}} = (-\sin\theta\, i + \cos\theta\, j)\dot{\theta}$

This implies $\dot{\boldsymbol{e}}_{\boldsymbol{r}} = \dot{\theta}\boldsymbol{e_\theta}$





$$\mathrm{v} = \dot{r}\boldsymbol{e_r} + r\dot{\boldsymbol{\theta}}\boldsymbol{e_r}$$

$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\boldsymbol{\theta}}\mathbf{e}_\theta$$

Thus, $\dot{r}$ is the radial component of the velocity vector, and $r\dot{\theta}$ is the transverse component.
To find the acceleration vector, we take the derivative of the velocity with respect to time. This gives

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{r}\mathbf{e}_r + \dot{r}\frac{d\mathbf{e}_r}{dt} + (\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta + r\dot{\theta}\frac{d\mathbf{e}_\theta}{dt}$$

The values of $\boldsymbol{de_r}/_{\boldsymbol{dt}}$ and $\boldsymbol{de_\theta}/_{\boldsymbol{dt}}$ are given by Equations

$$\dot{\boldsymbol{e}}_{\boldsymbol{r}} = \dot{\boldsymbol{\theta}}\boldsymbol{k} \times \boldsymbol{e}_{\boldsymbol{r}} = \dot{\boldsymbol{\theta}}\boldsymbol{e}_{\boldsymbol{\theta}}\ \boldsymbol{and}\ \dot{\boldsymbol{e}}_{\boldsymbol{\theta}} = \dot{\boldsymbol{\theta}}\boldsymbol{k} \times \boldsymbol{e}_{\boldsymbol{\theta}} = -\dot{\boldsymbol{\theta}}\boldsymbol{e}_{\boldsymbol{r}}$$

and yield the following equation for the acceleration vector in plane polar coordinates:

$$\boxed{\underset{\sim}{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\underset{\sim}{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\underset{\sim}{e}_\theta}$$

Thus, the radial component of the acceleration vector is

$$a_r = \ddot{r} - r\dot{\theta}^2$$

and the transverse component is

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$$

Module No. 86

Example of Radial and Transverse Components of Velocity and Acceleration

Statement:

On a horizontal turntable that is rotating at constant angular speed, a bug is crawling outward on a radial line such that its distance from the center increases quadratic ally with time: $r = bt^2$, $\theta = \omega t$, where b and ω are constants. Find the acceleration of the bug.

Solution:

The equation of acceleration in plane polar coordinates is given by

$$\boldsymbol{a} = \left(\ddot{\boldsymbol{r}} - \boldsymbol{r}\,\dot{\boldsymbol{\theta}}^2\right)\boldsymbol{e_r} + (\boldsymbol{r}\ddot{\boldsymbol{\theta}} + 2\dot{\boldsymbol{r}}\dot{\boldsymbol{\theta}})\boldsymbol{e_\theta} \tag{1}$$

As we have $r = bt^2 \; and \; \theta = \omega t$, therefore $\dot{r} = 2bt, \ddot{r} = 2b, \dot{\theta} = \omega \; and \; \ddot{\theta} = 0$.

By putting these values in equation (1), we have

$$\boldsymbol{a} = (2\boldsymbol{b} - \boldsymbol{bt^2\omega^2})\boldsymbol{e_r} + (\boldsymbol{bt^2}(0) + 2(2\boldsymbol{bt})\boldsymbol{\omega})\boldsymbol{e_\theta}$$

$$\boldsymbol{a} = \boldsymbol{b}(2 - \boldsymbol{t^2\omega^2})\boldsymbol{e_r} + 4\boldsymbol{bt\omega e_\theta}$$

Which is required acceleration of bug.

Module No. 87

Reference System or Inertial Frame

Framework

A framework that is used for the observation and mathematical description of physical phenomena and the formulation of physical laws, usually consisting of an observer, a coordinate system, and a clock or clocks assigning times at positions with respect to the coordinate system. a system of geometric axes in relation to which measurements of size, position, or motion can be made.

Inertial frames

A frame of reference in which a body remains at rest or moves with constant linear velocity unless acted upon by forces: any frame of reference that moves with constant velocity relative to an inertial system is itself an inertial system. These are also called inertial frame.
Mechanics is based on Newton's laws of motion. These laws are usually stated without explicitly discussing the role of frames of references. It is to be emphasized that Newton's laws are not valid w.r.t every frame of references e.g. they are not true in a rotating frame or coordinate system. They hold only in restricted class of frames, so called the inertial frames (Newtonian frames). Such frames are assumed to be un-accelerated i.e. neither moving in straight line with variable velocity nor rotating. In other words, it is supposed to be absolutely at rest (or moving with uniform velocity w.r.t absolutely-at-rest frames). We can easily see that if S is an inertial frame, then any other frame S' which is in uniform motion relative to S is also an inertial frame. To all observers in inertial frames, the force acting on a particle will be the same i.e. the law of motion will be invariant under transformation connecting inertial frames. In fact all laws of mechanics are same in all frames of reference. This statement is called Newtonian principal of relativity.

Module No. 88

Example of inertial Frame

In order to illustrate the concept of inertia, we will consider the example of earth.

Statement:

Calculate the centripetal acceleration relative to the acceleration due to gravity g, of

a) A point on the surface of the Earth's equator (the radius of the Earth is $R_e = 6.4 \times 10^6\ km$)

b) The Earth in its orbit about the Sun (the radius of the Earth's orbit is
   a. $a_e = 150 \times 10^6\ km$)

c) The Sun in its rotation about the center of the galaxy (the radius of the Sun's orbit about the center of the galaxy is $R_G = 2.8 \times 10^4\ LY$. its orbital speed is $v_G = 220\ km/s$)

Solution:

The centripetal acceleration of a point rotating in a circle of radius R is given by

$$a_c = \omega^2 R = \left(\frac{2\pi}{T}\right)^2 R = \frac{4\pi R^2}{T^2}$$

where T is period of one complete rotation ($T = 3.16 \times 10^7$). Thus, relative to g (gravitational acceleration), we have

$$\frac{a_c}{g} = \frac{4\pi^2 R^2}{gT^2} \qquad (1)$$

a) By putting value from (a) in (1) , we get

$$\frac{a_c}{g} = \frac{4(3.14)^2(6.4 \times 10^6)^2}{(9.8)(3.16 \times 10^7)^2}$$

$$\frac{a_c}{g} = 3.4 \times 10^{-3}$$

b) By putting values from (b) in (1), here

$$R = a_e \; = \; 150 \; \times \; 10^6 \; km$$

$$\frac{a_c}{g} = \frac{4(3.14)^2(150 \; \times \; 10^6)^2}{(9.8)(3.16 \times 10^7)^2}$$

$$\frac{a_c}{g} = 6 \; \times 10^4$$

c) By putting values from (c),

$$R = R_G \; = \; 2.8 \; \times \; 10^4 \; LY$$

$$\frac{a_c}{g} = \frac{4(3.14)^2(150 \; \times \; 10^6)^2}{(9.8)(2.8 \; \times \; 10^4 \;)^2}$$

$$\frac{a_c}{g} = 1.5 \; \times 10^{-12}$$

Module No. 89

Newton's First and Second Laws

Sir Issac Newton derives the Newton's laws of motion which are the three physical laws which together forms the foundation of classical mechanics.

Newton's 1st law of Motion

Newton's 1st law of motion is also called the law of inertia. This law measures the force of an object qualitatively.

We can state it as

"In the absence of an external force, an object continues its state of rest or motion with constant velocity."

The symbolic form of first law of motion is

$$\sum F = 0 \implies ma = 0$$

as the mass of the object is non-zero, therefore the acceleration of the concerned object must be zero

$$a = 0 \implies \frac{dv}{dt} = 0 \implies v = \text{constant}$$

As the consequence of first law of motion

- An object that is at rest will stay at rest unless a force acts upon it.
- An object continues its constant motion until an external force abrupt its state of motion.

The first law of motion says that if the net force acting on a body is zero then the velocity of object is said to be constant. The velocity of an object is vector quantity so if we say that the velocity of an object is constant we conclude that the speed and direction of the object is fixed.

Examples

Some of the examples of first law of motion are

i. A ball kicked in a ground.
ii. A car moving with constant velocity
iii. A book lying in a book shelf.

Newton's 2^{nd} law of Motion

Newton's 2^{nd} law of motion describes the relationship among the force, mass and acceleration of the given object.

We can states the 2^{nd} law of motion as

For any particle of mass m, the net force F on the particle of mass m times the particle's acceleration.

The symbolic form of first law of motion is

$$F = ma$$

The second law can be rephrased in terms of the particle's momentum, defined as

$$p = mv$$

by differentiating we get

$$\dot{p} = m\dot{v} = ma$$

hence we can state it as

$$F = \dot{p}$$

Consequently we can say the net force applied on a body is equals to the rate of change of momentum of the particle.

Examples

i. When we apply same force to move a truck and a bicycle, the bicycle will have more acceleration than the truck, because the mass of bicycle is less than the truck.

ii. An empty shopping cart is much easier to move than a full one, because the empty one has less mass.

Module No. 90

The Third Law and Law of Conservation of Momentum

Newton's first two laws concern the response of a single object to applied forces. The third law addresses a quite different issue: Every force on an object inevitably involves a second object the object that exerts the force. The nail is hit by the hammer; the cart is pulled by the horse, and so on. Newton realized that if an object 1 exerts a force on another object 2, then object 2 always exerts a force (the "reaction" force) back on object 1.

Newton's third law can be stated very compactly:

To every action there is an equal and opposite reaction.

The third law states that all forces between two objects exist in equal magnitude and opposite direction: if one object 1 exerts a force $\boldsymbol{F_{12}}$ on a second object *2*, then *S*imultaneously exerts a force $\boldsymbol{F_{21}}$ on *1*, and the two forces are equal in magnitude and opposite in direction: $\boldsymbol{F_{12} = -F_{21}}$. This law is sometimes referred to as the *action-reaction law*, with $\boldsymbol{F_{12}}$ called the "action" and $\boldsymbol{F_{21}}$ the "reaction".

In other situations the magnitude and directions of the forces are determined jointly by both bodies and it isn't necessary to identify one force as the "action" and the other as the "reaction". The action and the reaction are simultaneous, and it does not matter which is called the *action* and which is called *reaction*; both forces are part of a single interaction, and neither force exists without the other.

Examples

A variety of action-reaction force pairs are evident in nature.

i. A fish’s thrust through the water.
ii. A bird’s fly in the air.
iii. A rocket’s launch.
iv. The car moving on a road.
v. The nail hit by hammer.

Law of Conservation of Momentum

Definition of Linear Momentum

If a body of mass m is moving with velocity, then the momentum of that body is equals to the product of mass and velocity of the specific body.

Mathematically, we can write is as

$$\vec{p} = m\vec{v}$$

it is a vector quantity.

It S.I unit is $kg\square s^{-1}$.

Momentum of system of particles

The total momentum of a system of particles is the vector sum of the momenta of the individual particles:

$$\vec{p}_{systen} = \vec{p}_1 + \vec{p}_2 + \cdots \dots \dots + \vec{p}_n = \sum_{i=1}^{n} \vec{p}_i$$

where the system consists of n particles. we can also express it as

$$\vec{p}_{systen} = \sum_{i=1}^{n} \vec{p}_i = \sum_{i=1}^{n} m_i \vec{v}_i$$

where m_i is the mass of ith particle and $\vec{v}_i$ is corresponding velocity.

Statement “Law of conservation of momentum”

Law of conservation of momentum can be stated as:

If the sum of the external forces on a system is zero, the total momentum of the system does not change.

Mathematically, we can write is as

if

$$F_{ext} = 0 \Longrightarrow \vec{p} = 0 \Longrightarrow \sum_{i} m_i v_i = Mv = \text{constant}$$

Momentum is always conserved (even if forces are non-conservative).

As we stated earlier the second law of motion,

$$F_{ext} = ma$$

We can deduct the law of conservation of momentum from the last expression as well.

$$F_{ext} = 0 \Longrightarrow ma = 0 \Longrightarrow m\frac{dv}{dt} = 0$$

as mass is non-zero constant, therefore

$$\frac{dv}{dt} = 0 \text{ or } v = \text{constant} \Longrightarrow mv = \text{constant}.$$

Explanation

In simple terms, momentum is considered to be a quantity of motion. This quantity is measurable because if an object is moving and has mass, then it has momentum. Something that has a large mass has a large momentum or something that is moving very fast has a large momentum. The momentum of individual component may change but the total momentum of system remains conserved.

Example

i. A 3000 kg vehicle moving at 30 m/sec has a momentum of 90,000 kgm/sec as a result of product of the mass and the velocity.

ii. Two hockey players of equal mass are traveling towards each other, one is moving at 9 m/sec and the other at 5 m/sec. The one moving with the faster velocity has a greater momentum and will knock the other one backwards.

iii. A bullet fired from a gun, although small in mass, has a large momentum because of an extremely large velocity.

Module No. 91

Validity of Newton’s Law

- It is known that the Newton's laws are not valid universally after the quantum mechanics and theory of relativity.
- Initially Newton’s laws were considered as fundamental.
- But now quantum laws are fundamental and Newton’s laws can be observed as derived from the quantum world.
- So, we can say that

Quantum Mechanics

Classical Mechanics

- In the Classical phenomena, Newton’s first two laws are always valid.

Examples

- If the speed is extremely increased, even than first law remains valid.
- Newton’s second law $F = ma$ and its another form in terms of the particle's momentum $p = mv$ are no longer equivalent in relativity.

Module No. 92

Example of Newton's Law 1, 2

NEWTON'S 1st LAW

Example:

Because of the force field, a particle of mass 4 moves along a space curve C whose position vector is given as a function of time *t* by

$$r = (3t^3 + t^2)\hat{\imath} + (t^4 + 8t)\hat{\jmath} - 5t\hat{k}$$

Find

i. The velocity,
ii. The acceleration
iii. The momentum,
iv. The force field at any time t.

Solution:

i. Velocity= $v = \frac{dr}{dt}$

$$= \frac{d}{dt}((3t^3 + t^2)\hat{\imath} + (t^4 + 8t)\hat{\jmath} - 5t\hat{k})$$

$$= (9t^2 + 2t)\hat{\imath} + (4t^3 + 8)\hat{\jmath} - 5\hat{k}$$

ii. Acceleration= $a = \frac{dv}{dt} = \frac{d}{dt}((9t^2 + 2t)\hat{\imath} + (4t^3 + 8)\hat{\jmath} - 5\hat{k})$

$$= (18t + 2)\hat{\imath} + 12t^2\hat{\jmath}$$

iii. Momentum= $p = mv = 4((9t^2 + 2t)\hat{\imath} + (4t^3 + 8)\hat{\jmath} - 5\hat{k})$

$$= (36t^2 + 8t)\hat{\imath} + (16t^3 + 32)\hat{\jmath} - 20\hat{k}$$

iv. Force= $F = \frac{dp}{dt} = \frac{d}{dt}((36t^2 + 8t)\hat{\imath} + (16t^3 + 32)\hat{\jmath} - 20\hat{k})$

$$= (72t + 8)\hat{\imath} + 48t^2\hat{\jmath}$$

Newton's 2nd Law

Example:

A particle of mass 4 units moves in a force field given by

$$F = 4t^2\hat{\imath} + (16t - 4)\hat{\jmath} - 12t\hat{k}$$

Assume that initially, i.e at t = 0 the particle is located at $r_0 = i + 2j - 3k$ and has velocity $v_0 = i + 4j - 9k$, find

a) The velocity
b) The position at any time t.

Solution:

i. Since we know that

$$F = ma$$

As we have $m = 4$, therefore

$$F = 4a = 4\frac{dv}{dt} = 4t^2\hat{\imath} + (16t - 4)\hat{\jmath} - 12t\hat{k}$$

$$\frac{dv}{dt} = t^2\hat{\imath} + (4t - 1)\hat{\jmath} - 3t\hat{k}$$

Integrating the above equation gives the velocity

$$v = \frac{1}{3}t^3\hat{\imath} + (2t^2 - t)\hat{\jmath} + \frac{3}{2}t^2\hat{k} + C$$

where C is the constant of integration.

It is given that at $t = 0$, $v_0 = i + 4j - 9k$ at $t = 0$, therefore

$$i + 4j - 9k = C$$

So,

$$v = \frac{1}{3}t^3\hat{\imath} + (2t^2 - t)\hat{\jmath} + \frac{3}{2}t^2\hat{k} + i + 4j - 9k$$

$$v = (\frac{1}{3}t^3 + 1)\hat{\imath} + (2t^2 - t + 4)\hat{\jmath} + (\frac{3}{2}t^2 - 9)\hat{k}$$

ii. Since we know that

$$v = \frac{d\vec{r}}{dt} = (\frac{1}{3}t^3 + 1)\hat{\imath} + (2t^2 - t + 4)\hat{\jmath} + (\frac{3}{2}t^2 - 9)\hat{k}$$

There integrating w.r.to t gives the position

$$\vec{r} = \left(\frac{1}{12}t^4 + t\right)\hat{\imath} + \left(\frac{2}{3}t^3 - \frac{1}{2}t^2 + 4t\right)\hat{\jmath} + \left(\frac{1}{2}t^3 - 9t\right)\hat{k} + C'$$

As $r_0 = i + 2j - 3k$ at $t = 0$, we get $C' = i + 2j - 3k$

$$\vec{r} = \left(\frac{1}{12}t^4 + t\right)\hat{\imath} + \left(\frac{2}{3}t^3 - \frac{1}{2}t^2 + 4t\right)\hat{\jmath} + \left(\frac{1}{2}t^3 - 9t\right)\hat{k} + \ i + 2j - 3k$$

$$\vec{r} = \left(\frac{1}{12}t^4 + t + 1\right)\hat{\imath} + \left(\frac{2}{3}t^3 - \frac{1}{2}t^2 + 4t + 2\right)\hat{\jmath} + \left(\frac{1}{2}t^3 - 9t - 3\right)\hat{k}$$

Module No. 93

Example of Newton’s Law: 3

Example 1:

Determine the constant force needed to bring a 4000 lb mass moving at a speed of 90 ft/sec to rest in 3 sec?

Solution:

We know that the motion can be in any direction but for simplicity, we assume that the mass is moving in a straight axis. We have the information that initially the mass is moving, and finally it has to be at rest, therefore we can write as

$V_1 = 90$ ft./sec,

$V_2 = 0$ ft./sec,

where

$t = 4$ sec

Then using newton’s law $F = ma$

Where $a = \frac{V_2 - V_1}{t}$ (by using first equation of motion)

Now, $F = m\left(\frac{V_2 - V_1}{t}\right) = (4000)\left(\frac{0-90}{3}\right) = -120000 \text{ ft Ib/sec}^2$

This is important to note that there is a negative sign in the expression for F. This negative sign is because of the fact that the force is opposite to the direction of motion.

Example

A person of mass 85 kg is standing in a lift which is accelerating downwards at 0.45 ms^{-2} . Draw a diagram to show the forces acting on the person and calculate the force the person exerts on the floor of the lift. (© mathcentre 2009)

Solution

The weight of the person is 85 g, where g is the acceleration due to gravity.

where

$$W = 85\ g = 85 \times 9.81N$$

$$a = 0.45\ \mathrm{m}s^{-2}$$

The resultant force is W − R and using Newton's second law gives:

$$F = ma$$

$$W - R = F$$

$$W - R = 85 \times 0.45$$

$$R = W - 85 \times 0.45$$

$$R = 85 \times 9.81 - 85 \times 0.45$$

$$R = 795.6\ N = 800\ N$$

Therefore, using Newton's third law the force the person exerts on the floor of the lift is equal to the force of the floor acting on the person, i.e. R, which equals 800 N.

Module No. 94

Introduction to Energy: Kinetic Energy

Energy:

Energy is defined as the ability to do the work by the object. It is a measurable characteristic of a system which may be in the form of kinetic energy or potential.

There exist many forms of energy. The energy neither can be created nor be destroyed but can be converted from one form to another.

In mechanics, energy is the characteristic that transferred from one particle to another. The SI unit of energy is the joule; 1 joule can be defined as the energy transferred to an object by the work done of moving it a distance of 1 meter against a force of 1 newton.

The forms of energy include kinetic energy, potential energy, elastic energy, chemical energy, thermal energy and many others.

Kinetic energy is the energy stored in a body due to its motion. It can be transferred from one objects to another and transformed into other kinds of energy.

In classical mechanics, the kinetic energy is equal to 1/2 the product of the mass and the square of the speed. In formula form:

$$K.E = T = \frac{1}{2}mv^2$$

The measuring unit of kinetic energy is the joule.

It is denoted by T.

The kinetic energy increases with the square of the velocity. If a car is moving with double velocity then we can say that it has four times as much kinetic energy. As a consequence of this quadrupling, it takes four times the work to double the velocity.

if p denotes momentum of the object and m is the mass then we can symbolize the kinetic energy in the form of momentum as

$$T = \frac{p^2}{m}$$

Module No. 95

Introduction to Work – Theorem

When some external force is applied on an object, work is done by this force in the direction of force. Also when some work is done by the applied force, energy transferred from one place to another.

The work done can be defined as a product of force and the displacement in the direction of applied force. The amount of work done can be expressed as the following equation:

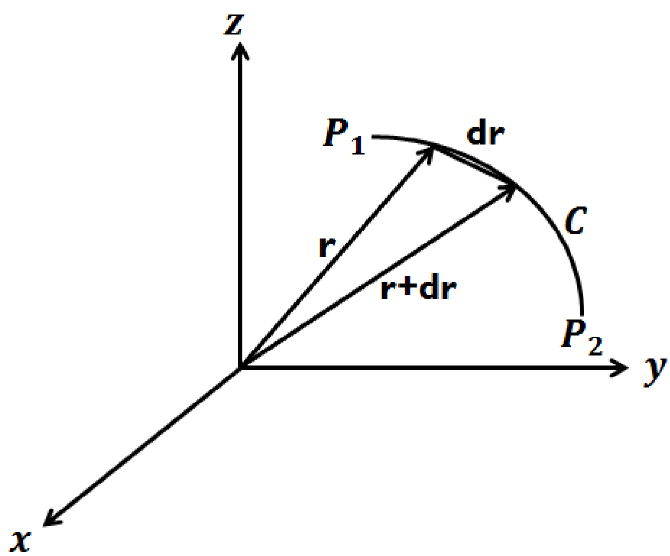
Work = Applied Force x Distance

The SI unit of work is the joule (J), which is defined as the work done by a force of one newton through a displacement of one meter.

If a force $\vec{F}$ acting on a particle gives it a displacement dr, then the work done by the force on the particle is defined as

$$dW = \vec{F}.dr$$

Since only the component of $\vec{F}$ in the direction of dr is effective in producing the motion.



The total work done by a force field (vector field) F in moving the particle from point P_1 to point P_2 along the curve C of Fig. is given by the line integral

$$W = \int_{P_1}^{P_2} \vec{F}.dr = \int_{r_1}^{r_2} \vec{F}.dr$$

Where r_1 and r_2 are the position vectors of P_1 and P_2 respectively.

Theorem Statement

A particle of constant mass m moves in space under the influence of a force field F. Assuming that at times t_1 and t_2 the velocity is $\vec{v}_1$ and $\vec{v}_2$ respectively, prove that the work done is the change in kinetic energy, i.e.,

$$\text{Work done} = \int_{t_1}^{t_2} \vec{F}.dr = \frac{1}{2}m\vec{v}_2^2 - \frac{1}{2}m\vec{v}_1^2$$

Proof

$$\text{L.H.S} = \text{Work done} = \int_{t_1}^{t_2} \vec{F}.dr = \int_{t_1}^{t_2} \vec{F}.\frac{dr}{dt}dt$$

$$= \int_{t_1}^{t_2} \vec{F}.\vec{v}\,dt$$

$$\therefore \frac{dr}{dt} = \vec{v}$$

$$= \int_{t_1}^{t_2} m\vec{a}.\vec{v}\,dt$$

$$\therefore F = ma$$

$$= m\int_{t_1}^{t_2} \frac{d\vec{v}}{dt}.\vec{v}\,dt = m\int_{t_1}^{t_2} \vec{v}.d\vec{v}$$

$$= \frac{1}{2}m\int_{t_1}^{t_2} d(\vec{v}.\vec{v}) = \frac{1}{2}m|\vec{v}^2|_{t_1}^{t_2}$$

$$\text{Work done} = \frac{1}{2}m\vec{v}_2^2 - \frac{1}{2}m\vec{v}_1^2$$

Hence the required equation.

Module No. 96

Example of Work Done

Example 1:

Find the work done in moving an object along a vector $r = 8\hat{\imath} + 2\hat{\jmath} - 5\hat{k}$ if the applied force is $F = 2\hat{\imath} - \hat{\jmath} - \hat{k}$.

Solution:

As we know the formula for the work done is

$$\boldsymbol{W} = \boldsymbol{F}.\boldsymbol{r}$$

$$= (2\hat{\imath} - \hat{\jmath} - \hat{k}).(8\hat{\imath} + 2\hat{\jmath} - 5\hat{k})$$

$$= 16 - 2 + 5 = 19\, joule$$

Work done = 19 joule

Example 2:

Find the work done by the force of a body of mass $m = 10^4 kg$ with initial velocity $v_1 = 1.5 \times 10^3 ms^{-1}$ to the final velocity $v_2 = 3.0 \times 10^3 ms^{-1}$ at any time t.

Solution:

As we studied the result that the work done is equals to the difference of kinetic energy of the initial and final points

$$\text{Work done} = \frac{1}{2}m\vec{v}_2^2 - \frac{1}{2}m\vec{v}_1^2$$

$$= \frac{1}{2}m(\vec{v}_2^2 - \vec{v}_1^2)$$

$$= \frac{1}{2}(10^4)[(3.0 \times 10^3)^2 - (1.5 \times 10^3)^2]$$

$$= 3.38 \times 10^{10} kgms^{-1}$$

$$= 3.38 \times 10^{10} N$$

Hence the required work done is $3.38 \times 10^{10} N$.

Module No. 97

Conservative Force Field

In order to understand the concept of conservative force field, we must understand the phenomena of path independence.

Path Independence

Independence of path is defined as that the work done by an object remains same between initial point and the final destination of the particle, it does not matter which path is taken by the object. The example of path independence is work done by the gravitation force. When an object freely falls from same height, the work done remains same no matter what path chose, when the object hits the ground due to the fixed gravitational force and the perpendicular distance between the object and the ground. The symbolic statement of path independence is

If a particle is moving along a curve C from $P_1$to P_2 is, then the total work done can be expressed as

$$W = \int_{P_1}^{P_2} \vec{F}.d\vec{r} = \int_{P_1}^{P_2} -(\nabla V).d\vec{r} = -|dV|_{P_1}^{P_2} = -[V(P_2) - V(P_1)] = V(P_1) - V(P_2)$$

In such case we can observe that the work done by the object depend only on the end points of the curve P_1 and P_2.

Now we can define the conservative force field in terms of path independence of a particle moving along a curve that:

"A force field is said to conservative if the total work done by the particle moving along a curve is independent of the path taken by the particle and depend upon the end points of the curve only."

Necessary and sufficient conditions for a Conservative Force Field

Conservative force fields conserve the following properties:

i. A force field F is conservative if and only if there exists a continuously differentiable scalar field V such that $\vec{F} = -\nabla V$ or, equivalently, if and only if

$$\text{curl } \vec{F} = \nabla \times \vec{F} = 0$$

identically

ii. A continuously differentiable force field F is conservative if and only if for any closed non-intersecting curve C (simple closed curve)

$$W = \oint_C \vec{F} \, . d\vec{r} = 0$$

i.e. the total work done in moving a particle around any closed path is zero.

Examples of Conservative Forces

- Gravitational force is an example of a conservative force.
- Elastic spring force is example of conservative force.
- The work done of a particle moving along a closed path is zero and the force which causes such motion is conservative.

Module No. 98

Example of Conservative Field

Problem:

Prove that the force field

$$\vec{F} = (y^2 - 2xyz^3)\hat{\imath} + (3 + 2xy - x^2z^3)\hat{\jmath} + (6z^3 - 3x^2yz^2)\hat{k}$$

is conservative.

Solution:

We know that he force field is conservative iff curl $\vec{F}$ is zero;

i.e. $\nabla \times \vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 - 2xyz^3 & 3 + 2xy - x^2z^3 & 6z^3 - 3x^2yz^2 \end{vmatrix}$$

$$= \hat{\imath}\left[\frac{\partial}{\partial y}(6z^3 - 3x^2yz^2) - \frac{\partial}{\partial z}(3 + 2xy - x^2z^3)\right] - \hat{\jmath}\left[\frac{\partial}{\partial x}(6z^3 - 3x^2yz^2) - \frac{\partial}{\partial z}(y^2 - 2xyz^3)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(3 + 2xy - x^2z^3) - \frac{\partial}{\partial y}(y^2 - 2xyz^3)\right]$$

$$= (-3x^2z^2 + 3x^2z^2)\hat{\imath} - (-6xyz^2 + -6xyz^2)\hat{\jmath} + (2y - 2xz^2 - 2y + 2xz^2)\hat{k}$$

$$= 0$$

As we obtain $\nabla \times \vec{F} = 0$

We conclude that $\vec{F}$ is conservative.

Module No. 99

Related Topic of Conservative Force Fields

Conservation of Energy Theorem in Case of Conservative Force Field

The concept of energy is a useful alternative to the concept of force. We will derive the law of conservation of energy using the concept of conservative force field.

If P_1 and P_2 are any two points on the trajectory of a particle, the total work done by the total force acting on the particle is given by

$$W = \int_{P_1}^{P_2} \vec{F}.d\vec{r} = \int_{t_1}^{t_2} \frac{d}{dt} mv.\frac{d\vec{r}}{dt} dt = \int_{t_1}^{t_2} m\frac{dv}{dt}.vdt$$

$$= \frac{1}{2}m \int_{t_1}^{t_2} \frac{d}{dt}(v^2)dt = \frac{1}{2}m|v^2|_{t_1}^{t_2} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 = T_2 - T_1 \qquad (1)$$

where $v(t_1) = v_1$ & $v(t_2) = v_2$

Now we suppose that the applied force F on the particle is conservative. Then we can express it as $F = -\nabla V$, and the integral for W can be written as

$$W = \int_{P_1}^{P_2} \vec{F}.d\vec{r} = \int_{P_1}^{P_2} -(\nabla V).d\vec{r} = -|dV|_{P_1}^{P_2} = -[V(P_2) - V(P_1)] = V_1 - V_2 \qquad (2)$$

where V_1 and V_2 corresponds to P.E. at initial and final positions of the particle. By comparing expression (1) and (2), we have

$$T_2 - T_1 = V_1 - V_2 \quad \text{or} \quad T_1 + V_1 = T_2 + V_2 \qquad (3)$$

Expression (3) shows that the quantity $T_1 + V_1$ is a constant of motion. It is called total energy of the particle. Denoting the total energy by E, we have

$$E = T + V = \frac{1}{2}mv^2 + V$$

Conservative Systems and Orbits of Particles

A single particle moving in a conservative field of forces may perform an important type of motion. Suppose the total energy E of the system is a constant of motion i.e.

$$\frac{1}{2}m\dot{r}^2 + V(r) = E$$

where E is some constant denoting total energy of the system.

Suppose the particle's motion in such that it returns to the same position, represented by the position vector r_0, at a later time. Then it must have the same K.E. and therefore the same speed. It follows that in a conservative system it is possible for closed trajectories to occur. This fact is very relevant in the study of Earth's motion about the Sun.

Module No. 100

Non-Conservative Force Field

Forces that cannot be expressed in the term of a potential energy function are called non-conservative forces. We can also state that forces that do not store energy are called non-conservative or dissipative forces. If there is no scalar function V such that $F = -\Delta V$ [or, equivalently, if $V \times F = 0$], then F is called a non-conservative force field.

Friction is a non-conservative force, and there are others. It is always opposed to the direction of motion and is not a single valued function of position alone.

Similarly the impulse (time dependent force) is also non-conservative and cannot be derived from a scalar point function.

An example of non-conservative force, we have $F = kv$, where v is the velocity of the particle, then,

$$\oint \vec{F}.d\vec{r} = \int \vec{F}.\frac{d\vec{r}}{dt}dt$$

$$= \int_{t_1}^{t_2} \vec{F}.\vec{v}dt = k\int_{t_1}^{t_2} v^2dt > 0$$

which shows the integral is not equals to zero. Hence the force is non-conservative.

Work-Energy relation and Non-conservative Forces

We have already shown that for any general force F

$$W = \int_{P_1}^{P_2} \vec{F}.d\vec{r} = T_2 - T_1$$

When the force F can be broken into conservative and non-conservative parts

$$\vec{F} = \vec{F}^{(c)} + \vec{F}^{(nc)}$$

we have

$$\int_{P_1}^{P_2} \vec{F}^{(c)}.d\vec{r} + \int_{P_1}^{P_2} \vec{F}^{(nc)}.d\vec{r} = T_2 - T_1 \qquad (1)$$

But $\vec{F}^{(c)} = -\nabla V$ and equation (1) can be written as

$$V_1 - V_2 + \int_{P_1}^{P_2} \vec{F}^{(nc)}.d\vec{r} = T_2 - T_1$$

which can also be written as

$$T_1 + V_1 + \int_{P_1}^{P_2} \vec{F}^{(nc)}.d\vec{r} = T_2 + V_2 \tag{2}$$

The work done in overcoming friction is always negative, because $\vec{F}^{(nc)}$ is opposite to the displacement relation (2) proves that the influence of friction is dissipative and therefore decrease the total mechanical energy of the system.

Alternatively (2) can be expressed as

$$(V_2 - V_1) + (T_2 - T_1) = \int_{P_1}^{P_2} \vec{F}^{(nc)}.d\vec{r}$$

or

$$\Delta(V + T) = \int_{P_1}^{P_2} \vec{F}^{(nc)}.d\vec{r}$$

It is interesting to remember that the process in which work is converted into internal energy (due to friction) are irreversible.