Lecture Handouts

on

VECTORS AND CLASSICAL MECHANICS (MTH-622)

Virtual University of Pakistan
Pakistan

About the Handouts

The following books have been mainly followed to prepare the slides and handouts:

1. Spiegel, M.R., *Theory and Problems of Vector Analysis: And an Introduction to Tensor Analysis*. 1959: McGraw-Hill.
2. Spiegel, M.S., *Theory and problems of theoretical mechanics*. 1967: Schaum.
3. Taylor, J.R., *Classical Mechanics*. 2005: University Science Books.
4. DiBenedetto, E., *Classical Mechanics: Theory and Mathematical Modeling*. 2010: Birkhäuser Boston.
5. Fowles, G.R. and G.L. Cassiday, *Analytical Mechanics*. 2005: Thomson Brooks/Cole.

The first two books were considered as main text books. Therefore the students are advised to read the first two books in addition to these handouts. In addition to the above mentioned books, some other reference book and material was used to get these handouts prepared.

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Module No. 51

Selected Example/Problem 2: Volume Integral

Problem Statement

If $\vec{F} = (2x^2 - 3z)\hat{\imath} - 2xy\hat{\jmath} - 4x\hat{k}$, evaluate

$$\iiint\limits_R \nabla.\vec{F}dV$$

the closed region bounded by the planes $x = 0, \mathrm{y} = 0, z = 0$ and $2x + 2\mathrm{y} + z = 4$.

Solution

Since $\vec{F} = (2x^2 - 3z)\hat{\imath} - 2xy\hat{\jmath} - 4x\hat{k},$

Thus

$$\nabla.\vec{F} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).\left((2x^2 - 3z)\hat{\imath} - 2xy\hat{\jmath} - 4x\hat{k}\right)$$

$$= \frac{\partial(2x^2 - 3z)}{\partial x} - \frac{\partial 2xy}{\partial y} - \frac{\partial 4x}{\partial z}$$

$$= 4x - 2x = 2x$$

$$\iiint\limits_R \nabla.\vec{F}dV = \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} 2xdzdydx$$

Using equation $2x + 2\mathrm{y} + z = 4$ for limit

Integrating w.r.t z we obtain,

$$= 2\int_0^2 \int_0^{2-x} x|z|_0^{4-2x-2y}dydx = 2\int_0^2 \int_0^{2-x} x(4 - 2x - 2y)dydx = \int_0^2 \int_0^{2-x} (8x - 4x^2 - 4xy)dydx$$

$$= \int_0^2 (8x|y|_0^{2-x} - 4x^2|y|_0^{2-x} - 4x\left|\frac{y^2}{2}\right|_0^{2-x})dx = \int_0^2 8x(2-x) - 4x^2(2-x) - 2x(2-x)^2dx$$

$$= \int_0^2 8x - 8x^2 + 2x^3 dx$$

$$= \left| 4x^2 - \frac{8}{3}x^3 + \frac{x^4}{2} \right|_0^2$$

$$= 16 - \frac{64}{3} + 8 = -\frac{8}{3}$$

Module No. 52

Divergence Theorem

Divergence theorem is also called Gauss's divergence theorem. Gauss's divergence theorem has wide applications in physics and engineering and is used to derive equation governing the flow of fluids, heat conduction, wave propagation and electrical fields.

In words the divergence theorem may states that the surface integral of the normal component of a vector function $\vec{A}$ taken over a closed surface S is equal to the integral of the divergence of $\vec{A}$ taken over the region R enclosed by the surface.

We can write it mathematically as

Statement

It states that if R is the region bounded by a closed surface S and $\vec{A}$ is a vector point function with continuos first partia derivatives, then

$$\iint_S \vec{\boldsymbol{A}}.\hat{\boldsymbol{n}}\boldsymbol{dS} = \iiint_R \boldsymbol{\nabla}.\vec{\boldsymbol{A}}\boldsymbol{dV}$$

Where $\hat{n}$ is outward drawn unit normal to S.

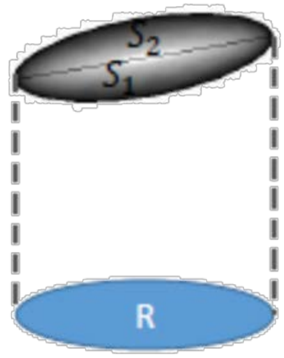
Proof

If $\vec{A}$ is expressed as $\vec{A} = A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k},$ then the divergence theorem can be written component wise as

$$\iint_S (A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k}).\hat{n}dS = \iiint_R (\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z})dV$$

To establish this relation, we will prove that the respective integrals on each sides are equal.

We prove this for a closed surface S, which has the property that any line parallel to the coordinate axes cuts S in at most two points. Under this assumption, it follows that S is doubled valued surface over its projection on each of the coordinate planes.



Let R' be the projection of S on the xy-plane. Divide the surface S into the lower and upper parts S_1 and S_2 and assume the equations of S_1 and S_2 to be $z = f_1(x, y)$ and $z = f_2(x, y)$ respectively. Consider

$$\iiint_R \frac{\partial A_3}{\partial z} dV = \iiint_R \frac{\partial A_3}{\partial z} dzdydx$$

$$= \iint_{R'} \left[\int_{z=f_1(x,y)}^{z=f_2(x,y)} \frac{\partial A_3}{\partial z} dz \right] dydx = \iint_{R'} |A_3(x, y, z)|_{z=f_1(x,y)}^{z=f_2(x,y)} dydx$$

$$= \iint_{R'} \{A_3[(x, y, f_2(x, y))] - A_3[(x, y, f_1(x, y))]\} dydx \qquad (1)$$

For the upper part $S_2 = dydx = \cos\gamma_2 dS_2 = \hat{k}.\hat{n}_2 dS_2$, since the normal $\hat{n}$ to S_2 *makes an acute angle* with k. For the lower part S_1, $dydx = -\cos\gamma_1 dS_2 = \hat{k}.\hat{n}_1 dS_1$, since the normal $\hat{n}_1$ to S_1, makes an angle γ_1 with $-\hat{k}$.

Then

$$\iint_{R'} A_3\big[\big(x, y, f_2(x, y)\big)\big] = \iint_{S_2} A_3\, \hat{k}.\hat{n}_2 dS_2$$

And

$$\iint_{R'} A_3\big[\big(x, y, f_1(x, y)\big)\big] = -\iint_{S_1} A_3\, \hat{k}.\hat{n}_1 d\square_1$$

and therefore the equation (1) becomes

$$\iiint\limits_R \frac{\partial A_3}{\partial z}\,dV = \iint\limits_{S_2} A_3\,\hat{k}.\hat{n}_2 dS_2 + \iint\limits_{S_1} A_3\,\hat{k}.\hat{n}_1 dS_1$$

$$\iiint\limits_R \frac{\partial A_3}{\partial z}\,dV = \iint\limits_S A_3\,\hat{k}.\hat{n} dS \qquad (2)$$

Similarly, by projecting S on the yz and zx coordinate plane, we obtain respectively,

$$\iiint\limits_R \frac{\partial A_1}{\partial z}\,dV = \iint\limits_S A_1\,\hat{\imath}.\hat{n} dS \qquad (3)$$

$$\iiint\limits_R \frac{\partial A_2}{\partial z}\,dV = \iint\limits_S A_2\,\hat{\jmath}.\hat{n} dS \qquad (4)$$

By adding equation (2),(3) and (4), we obtain

$$\iiint\limits_R \frac{\partial A_1}{\partial z}\,dV + \iiint\limits_R \frac{\partial A_2}{\partial z}\,dV + \iiint\limits_R \frac{\partial A_3}{\partial z}\,dV = \iint\limits_S A_1\,\hat{\imath}.\hat{n} dS + \iint\limits_S A_2\,\hat{\jmath}.\hat{n} dS + \iint\limits_S A_3\,\hat{k}.\hat{n} dS$$

Which is equal to

$$\iiint\limits_R \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) dV = \iint\limits_S (A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k}).\hat{n} dS$$

or

$$\iint\limits_S (A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k}).\hat{n} dS = \iiint\limits_R \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) dV$$

Hence the theorem.

Module No. 53

Divergence theorem in Rectangular Form

As we studies earlier about the rectangular coordinate system and rectangular coordinates. The Cartesian coordinate system (x, y, z) is also called rectangular system. In this article, we will express the gauss's divergence theorem in the form of Cartesian coordinates.

Let $\vec{A} = A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k}$, and $\hat{n} = n_1\hat{\imath} + n_2\hat{\jmath} + n_3\hat{k}$

Then

$$\nabla.\vec{A} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).\left(A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k}\right)$$

$$= \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

and

$$\vec{A}.\hat{n} = \left(A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k}\right).\left(n_1\hat{\imath} + n_2\hat{\jmath} + n_3\hat{k}\right)$$

The unit normal to S is $\hat{n} = n_1\hat{\imath} + n_2\hat{\jmath} + n_3\hat{k}$. Then $\hat{n}\,.\,\hat{\imath} = n_1 = cos\ \alpha$, $\hat{n}\,.\,\hat{\jmath} = n_2 = cos\ \beta$ and $\hat{n}\,.\,\hat{k} = n_3 = cos\ \gamma$, where α, β, γ are the angles which $\hat{n}$ makes with the positive x, y, z-axes or $\hat{\imath}, \hat{\jmath}, \hat{k}$ directions respectively. The quantities $cos\ \alpha$, $cos\ \beta$, $cos\ \gamma$ are the direction cosines of $\hat{n}$.

Hence

$$\vec{A}.\hat{n} = A_1 \cos\alpha + A_2 \cos\beta + A_3 \cos\gamma$$

Using these values, Gauss's divergence theorem can be written as

$$\iiint_R \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)dV = \iint_S (A_1 \cos\alpha + A_2 \cos\beta + A_3 \cos\gamma)dS$$

is the required form of divergence theorem.

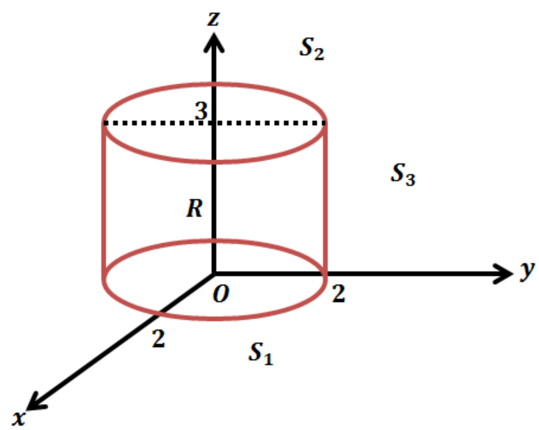
Module No. 54

Verification of Divergence Theorem by an Example

Problem Statement

Verify the divergence theorem for $\vec{A} = 4x\hat{\imath} — 2y^2\hat{\jmath} + z^2\hat{k}$ taken over the region bounded by $x^2 + y^2 = 4, z = 0$ and $z = 3$.

Solution



[1]

As we know the divergence theorem is

$$\iint_S \vec{A}.\hat{n}dS = \iiint_V \nabla.\vec{A}dV$$

$$\iiint_V \nabla.\vec{A}dV = \iiint_V \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)dV$$

$$= \iiint_V \left(\frac{\partial 4x}{\partial x} - \frac{\partial\, 2y^2}{\partial y} + \frac{\partial z^2}{\partial z}\right)dV = \iiint_V (4 - 4y + 2z)dV$$

$$= \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{z=0}^{3} (4 - 4y + 2z)dzdydx$$

Here we use the given equation $x^2 + y^2 = 4$, for limits

The surface S of the cylinder consists of a base S_1 $(r = 0)$, the top S_2 $(r = 3)$ and the convex portion $S_3(x^2 + y^2 = 4)$. Then

$$\text{Surface Integral} = \iint\limits_S \vec{A}.\hat{n}dS = \iint\limits_{S_1} \vec{A}.\hat{n}dS_1 + \iint\limits_{S_2} \vec{A}.\hat{n}dS_2 + \iint\limits_{S_3} \vec{A}.\hat{n}dS_3$$

On Surface S_1 $(z = 0)$, $\hat{n} = -\hat{k}, \vec{A} = 4x\hat{\imath} - 2y^2\hat{\jmath}$

Therefore $\vec{A}.\hat{n} = \left(4x\hat{\imath} - 2y^2\hat{\jmath}\right).\left(-\hat{k}\right) = 0$

$$\Longrightarrow \iint\limits_{S_1} \vec{A}.\hat{n}dS_1 = 0$$

On Surface S_2 $(z = 3)$, $\hat{n} = \hat{k}, \vec{A} = 4x\hat{\imath} - 2y^2\hat{\jmath} + 9\hat{k}$

Therefore $\vec{A}.\hat{n} = \left(4x\hat{\imath} - 2y^2\hat{\jmath} + 9\hat{k}\right).\left(\hat{k}\right) = 9$

$$\Longrightarrow \iint\limits_{S_2} \vec{A}.\hat{n}dS_2 = 9\iint\limits_{S_2} dS_2 = 9(4\pi) = 36\pi$$

Since we have $x^2 + y^2 = 4 \Longrightarrow \sqrt{x^2 + y^2} = 2 = r$, radius of the base of the cylinder.

Therefore the area of the base of cylinder is $\pi r^2 = 4\pi$

On S_3 $(x^2 + y^2 = 4)$. A perpendicular to $x^2 + y^2 = 4$ has the direction

$$\nabla(x^2 + y^2) = 2x\hat{\imath} + 2y\hat{\jmath}$$

Then a unit normal is

$$\hat{n} = \frac{2x\hat{\imath} + 2y\hat{\jmath}}{\sqrt{4x^2 + 4y^2}} = \frac{2(x\hat{\imath} + y\hat{\jmath})}{2\sqrt{x^2 + y^2}} = \frac{(x\hat{\imath} + y\hat{\jmath})}{\sqrt{4}} = \frac{(x\hat{\imath} + y\hat{\jmath})}{2}$$

Since $x^2 + y^2 = 4$

$$\vec{A}.\hat{n} = \left(4x\hat{\imath} - 2y^2\hat{\jmath} + z^2\hat{k}\right).\frac{(x\hat{\imath} + y\hat{\jmath})}{2} = 2x^2 - y^3$$

$$\iint\limits_{S_3} (2x^2 - y^3)dS_3$$

Since $r = 2$ is the radius of the base of the cylinder

So, using cylindrical coordinates, $x = 2\cos\theta, y = 2\sin\theta\,, dS_3 = 2d\theta dz$

We have

$$\iint\limits_{S_3} (2x^2 - y^3)dS_3 = \int\limits_{\theta=0}^{2\pi} \int\limits_{z=0}^{3} [2(2\cos\theta)^2 - (2\sin\theta)^3]2dzd\theta$$

$$= \int\limits_{0}^{2\pi} 48\cos^2\theta - 48\sin^2\theta \, d\theta = 48\pi$$

Then the surface integral= $0 + 36\pi + 48\pi = 84\pi$, agreeing with the volume integral and verifying the divergence theorem.

Module No. 55

Another Example: Divergence Theorem

Problem Statement

Evaluate

$$\iiint_R \vec{F}.\hat{n}dV$$

where $F = 4xz\hat{\imath} — y^2\hat{\jmath} + yz\hat{k}$ and S is the surface of the cube bounded S by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1.$

Solution

By the divergence theorem, the required integral is equal to

$$\iint_S \vec{A}.\hat{n}dS = \iiint_R \nabla.\vec{A}dV$$

$$\nabla.\vec{A} = \nabla.\left(4xz\hat{\imath} — y^2\hat{\jmath} + yz\hat{k}\right)$$

$$= \frac{\partial 4xz}{\partial x} - \frac{\partial y^2}{\partial y} + \frac{\partial yz}{\partial z}$$

$$\iiint_R \nabla.\vec{A}dV = \iiint_R \left(\frac{\partial 4xz}{\partial x} - \frac{\partial y^2}{\partial y} + \frac{\partial yz}{\partial z}\right)dV$$

Now

$$= \iiint_R (4z - 2y + y)dV = \iiint_R (4z - y)dV$$

$$= \int_0^1 \int_0^1 \int_0^1 (4z - y)dzdydx$$

$$= \int_0^1 \int_0^1 2z^2 - yzdydx = \int_0^1 \int_0^1 2 - ydydx$$

$$= \int_0^1 2y - \frac{y^2}{2} dx = \int_0^1 2 - \frac{1}{2} dx = \frac{3}{2}\int_0^1 dx = \frac{3}{2}|x|_0^1 = \frac{3}{2}$$

Hence

$$\iiint_R \vec{F}.\hat{n} dV = \frac{3}{2}$$

Module No. 56

Further Example 1 of Divergence Theorem

Problem Statement

Prove the identity

$$\iiint_V \nabla\varphi dV = \iint_S \varphi\hat{n} dS$$

Proof

In the divergence theorem, let $\vec{A} = \varphi\vec{C}$ where $\vec{C}$ a constant vector is. Then

$$\iiint_V \nabla.(\varphi\vec{C})dV = \iint_S \varphi\vec{C}.\hat{n} dS$$

Since $\nabla.\left(\varphi\vec{C}\right) = (\nabla\varphi).\vec{C}) = \vec{C}.\nabla\varphi$ and $\varphi\vec{C}.\hat{n} = \vec{C}.(\varphi\hat{n})$

Substituting these values in above integral, we get

$$\iiint_V \vec{C}.\nabla\varphi dV = \iint_S \vec{C}.(\varphi\hat{n}) dS$$

Taking $\vec{C}$ outside the integrals,

$$\vec{C}.\iiint_V \nabla\varphi dV = \vec{C}.\iint_S (\varphi\hat{n}) dS$$

and since C is an arbitrary constant vector,

$$\iiint_V \nabla\varphi dV = \iint_S (\varphi\hat{n}) dS$$

Hence the result.

Module No. 57

Further Example 2 of Divergence Theorem

A fluid of density $\rho(x, y, z, t)$ moves with velocity $v(x, y, z, t)$. If there are no sources or sinks, prove that

$$\nabla. J + \frac{\partial \rho}{\partial t} = 0$$

Solution

Consider an arbitrary surface enclosing a volume V of the fluid. At any time the mass of fluid within V is

$$M = \iiint_V \rho dV$$

The time rate of increase of this mass is

$$\frac{\partial M}{\partial t} = \frac{\partial}{\partial t} \iiint_V \rho dV = \iiint_V \frac{\partial \rho}{\partial t} dV$$

Let A = velocity v at any point of a moving fluid

Volume of fluid crossing dS in Δt seconds = volume contained in cylinder of base dS and slant height $v\Delta t$

$$= (v\Delta t).\hat{n}dS = v.\hat{n}\, dS\, \Delta t$$

Then, volume per second of fluid crossing $dS = v.\hat{n}dS$

The relation of mass of fluid per unit time leaving V is

$$\iint_S \rho v.\hat{n}dS$$

and the time rate of increase in mass is

$$-\iint_S \rho v.\hat{n}dS = \iiint_V \nabla. \rho v dV$$

by the divergence theorem. Then

$$\iiint\limits_V \frac{\partial\rho}{\partial t} dV = -\iiint\limits_V \nabla.\rho v d\square$$

$$\iiint\limits_V \frac{\partial\rho}{\partial t} + \nabla.(\rho v) dV = 0$$

Suppose that $\iiint_V \frac{\partial\rho}{\partial t} + \nabla.(\rho v)dV = 0$ for all the region V. If we suppose that $\frac{\partial\rho}{\partial t} + \nabla.(\rho v)dV > 0$ at a point P, then from the continuity of the derivatives it follows that $\frac{\partial\rho}{\partial t} + \nabla.(\rho v)dV > 0$ in some region A surrounding P. If Γ is the boundary of A then

$\iiint_V \frac{\partial\rho}{\partial t} + \nabla.(\rho v)dV > 0$ which contradicts the assumption that the line integral is zero around every closed curve. Similarly the assumption $\frac{\partial\rho}{\partial t} + \nabla.(\rho v)dV < 0$ leads to a contradiction. Hence the integrand $\frac{\partial\rho}{\partial t} + \nabla.(\rho v)dV$ must be equal to zer0.

Hence from the continuity of the derivatives it follows that

$$\nabla.J + \frac{\partial\rho}{\partial t} = 0$$

Where $J = \rho v$.

The equation is called the continuity equation. If ρ is a constant, the fluid is incompressible and $\nabla.v = 0$, i.e. v is solenoidal.

The continuity equation also arises in electromagnetic theory, where ρ is the charge density and $J = \rho v$ is the current density.

Module No. 58

Further Example 3 of Divergence Theorem

Problem Statement

Prove the relation:

$$\iiint_V \nabla \times \vec{B} dV = \iint_S \hat{n} \times \vec{B}\, dS$$

where $\vec{B}$ is any vector field.

Proof

Since the divergence theorem,

$$\iiint_V \nabla . \vec{A} dV = \iint_S \vec{A} . \hat{n} dS$$

let $\vec{A} = \vec{B} \times \vec{C}$ where $\vec{C}$ is a constant vector. Then

$$\iiint_V \nabla . (\vec{B} \times \vec{C}) dV = \iint_S (\vec{B} \times \vec{C}) . \hat{n} dS$$

Since

$\nabla . (\vec{B} \times \vec{C}) = \vec{C} . (\nabla \times \vec{B})$ and $(\vec{B} \times \vec{C}) . \hat{n} = \vec{B} . (\vec{C} \times \hat{n}) = (\vec{C} \times \hat{n}) . \vec{B} = \vec{C} . (\hat{n} \times \vec{B})$,

Then the above integral will become

$$\iiint_V \vec{C} . (\nabla \times \vec{B}) dV = \iint_S \vec{C} . (\hat{n} \times \vec{B}) dS$$

Taking $\vec{C}$ outside the integral,

$$\vec{C} . \iiint_V (\nabla \times \vec{B}) dV = \vec{C} . \iint_S (\hat{n} \times \vec{B}) dS$$

and since $\vec{C}$ Is an arbitrary constant vector,

$$\iiint_V \nabla \times \vec{B}\,dV = \iint_S \hat{n} \times \vec{B}\,dS$$

Hence the result.

Module No. 59

Stokes' Theorem

In words we can state Stokes' theorem as the line integral of the tangential component of a vector function $\vec{A}$ taken around a simple closed curve C is equal to the surface integral of the normal component of the curl of $\vec{A}$ taken over any surface S having C as its boundary.

Statement

It states that if S is an open, two sided surface bounded by a simple closed curve C, then if $\vec{A}$ has continuous first partial derivatives

$$\oint_C \vec{A}.d\vec{r} = \iint_S (\nabla \times \vec{A}).\hat{n}dS$$

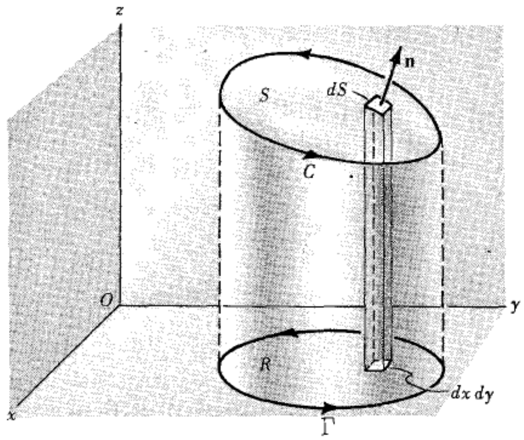
Where C is traversed in the positive direction.

Proof

If $\vec{A}$ is expressed as $\vec{A} = A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k}$, then the divergence theorem can be written as

$$\iint_S \nabla \times \left(A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k}\right)dS = \oint_C A_1dx + A_2dy + A_3dz$$

We will prove this theorem for a surface S which has the property that its projection on the xy, yz and zx planes are regions bounded by simple closed curves as shown in figure.



[1]

Assume S to have representation $z = f(x,y)$ or $x = g(y,z)$ or $y = h(x,z)$, where f, g, h are continuous and differentiable functions.

Consider first

$$\iint_S \; [\nabla \times (A_1\hat{\imath})] \, . \, \hat{n} dS$$

Since,

$$\nabla \times (A_1\hat{\imath}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_1 & 0 & 0 \end{vmatrix} = \left(\frac{\partial A_1}{\partial yz}\hat{\jmath} - \frac{\partial A_1}{\partial y}\hat{k}\right)$$

Therefore,

$$[\nabla \times (A_1\hat{\imath})]. \hat{n} dS = \left(\frac{\partial A_1}{\partial z}\hat{n}.\hat{\jmath} - \frac{\partial A_1}{\partial y}\hat{n}.\hat{k}\right) dS \qquad (1)$$

If $z = f(x,y)$is taken the equation of S, then the position vector to any point of S is

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} = x\hat{\imath} + y\hat{\jmath} + f(x,y)\hat{k}$$

so that

$$\frac{\partial \vec{r}}{\partial y} = \hat{\jmath} + \frac{\partial z}{\partial y}\hat{k} = \hat{\jmath} + \frac{\partial f}{\partial y}\hat{k}$$

But $\frac{\partial \vec{r}}{\partial y}$ is a vector tangent to S and thus perpendicular to $\hat{n}$, so that

$$\hat{n}.\frac{\partial \vec{r}}{\partial y} = \hat{n}.\hat{\jmath} + \frac{\partial z}{\partial y}\hat{n}.\hat{k} = 0$$

or we can write

$$\hat{n}.\hat{j} = -\frac{\partial z}{\partial y}\hat{n}.\hat{k}$$

Substituting this value in equation (1), we obtain

$$[\nabla \times (A_1\hat{\imath})].\hat{n}dS = \left(-\frac{\partial A_1}{\partial z}\frac{\partial z}{\partial y}\hat{n}.\hat{k} - \frac{\partial A_1}{\partial y}\hat{n}.\hat{k}\right)dS$$

$$= -\left(\frac{\partial A_1}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial A_1}{\partial y}\right)\hat{n}.\hat{k}dS \qquad (2)$$

Now on S, $A_1(x, y, z) = \big(x, y, f(x, y)\big) = F(x, y)$ (3)

Hence equation $\frac{\partial A_1}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial A_1}{\partial y} = \frac{\partial F}{\partial y}$ and (2) becomes

$$[\nabla \times (A_1\hat{\imath})].\hat{n}dS = -\frac{\partial F}{\partial y}\hat{n}.\hat{k}dS = -\frac{\partial F}{\partial y}dxdy$$

Then

$$\iint_S [\nabla \times (A_1\hat{\imath})].\hat{n}dS = \iint_R -\frac{\partial F}{\partial y}dxdy$$

where R is the projection of S on the xy-plane.

By the Green's theorem in the plane, the last integral equals $\oint_\Gamma Fdx$ where Γ is the boundary of R. From equation (3), since at each point (x, y) of Γ the value of F is the same as the value of A_1 at each point (x, y, z) of C, and since dx is the same for both curves, we must have

$$\oint_\Gamma Fdx = \oint_C A_1dx$$

or

$$\iint_S [\nabla \times (A_1\hat{\imath})].\hat{n}dS = \oint_C A_1dx \qquad (4)$$

Similarly, by projections on the other coordinate planes, we have

$$\iint_S [\nabla \times (A_2\hat{\jmath})].\hat{n}dS = \oint_C A_2dy \qquad (5)$$

$$\iint_S \big[\nabla \times \big(A_3\hat{k}\big)\big].\hat{n}dS = \oint_C A_3dz \qquad (6)$$

Addition of equation (4), (5) and (6) gives us the required results and completes the theorem.

The theorem is also valid for surfaces S which may not satisfy the restrictions imposed above. For assume that S can be subdivided into surfaces $S_{1,}S_{2,}S_{3,} \dots \dots . . S_k$ with boundaries $C_{1,}C_{2,}C_{3,} \dots \dots . . C_k$ which do satisfy the restrictions. Then Stokes' theorem holds for each such surface. Adding these surface integrals, the total surface integral over S is obtained. Adding the corresponding line integrals over $C_{1,}C_{2,}C_{3,} \dots \dots . . C_k$, the line integral over C is obtained.

Module No. 60

Stokes' Theorem in Rectangular Form

Let $\vec{A} = A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k}$ and $\hat{n} = n_1\hat{\imath} + n_2\hat{\jmath} + n_3\hat{k}$ be the outward drawn unit normal to the surface S. If α, β and γ are the angles which the unit normal $\hat{n}$ makes with the positive directions of x, y and z axes respectively, then

$n_1 = \hat{n}.\hat{\imath} = \cos\alpha\,, n_2 = \hat{n}.\hat{\jmath} = \cos\beta$ and $n_3 = \hat{n}.\hat{k} = \cos\gamma.$

The quantities $\cos\alpha\,, \cos\beta,$ and $\cos\gamma$ are the direction cosine of $\hat{n}$. Then

$$\hat{n} = \cos\alpha\,\hat{\imath} + \cos\beta\,\hat{\jmath} + \cos\gamma\,\hat{k}$$

Thus

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\hat{\imath} + \left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\hat{k}$$

and

$$(\nabla \times \vec{A}).\hat{n}$$

$$= \left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\hat{\imath} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\hat{k}\right].(\cos\alpha\,\hat{\imath} + \cos\beta\,\hat{\jmath} + \cos\gamma\,\hat{k})$$

$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial A_1}{\partial z} - \frac{\square A_3}{\partial x}\right)\cos\beta + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\cos\gamma$$

Also

$$\vec{A}.d\vec{r} = \left(A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k}\,\right).(dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k})$$

$$= A_1dx + A_2dy + A_3dz$$

and the stokes theorem becomes

$$\oint_C A_1 dx + A_2 dy + A_3 dz$$

$$= \iint_S \left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\cos\beta + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\cos\gamma\right] dS$$

Module No. 61

Verification of Stokes' Theorem by an Example

Problem Statement

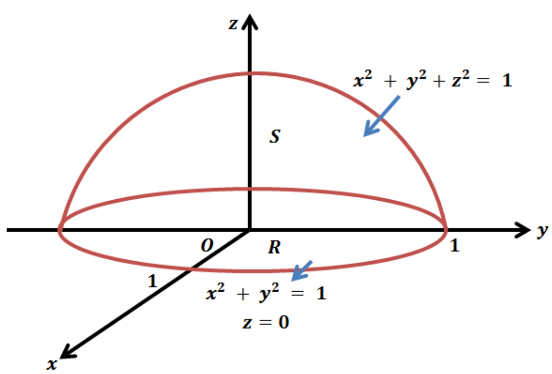
Verify Stokes' theorem for $\vec{A} = (2x - y)\hat{\imath} - yz^2\hat{\jmath} - y^2z\hat{k}$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary.

Solution

The stokes' theorem is

$$\oint_C \vec{A}.d\vec{r} = \iint_S (\nabla \times \vec{A}).\hat{n}dS$$

we will verify the above statement using given vector function $\vec{A}$.



[1]

Let we solve first $\oint_C \vec{A}.d\vec{r}$

The boundary C of S is a circle in the xy plane of radius one and center at the origin. Let $x = cost$,$y = sint$, $z = 0, 0 < t < 2\pi$ be parametric equations of C.(since $r = 1$)

$$\oint_C \vec{A}.d\vec{r} = \oint_C (2x - y)\hat{\imath} - yz^2\hat{\jmath} - y^2z\hat{k}).(dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k})$$

$$= \oint_C (2x - y)dx - yz^2dy - y^2zdz$$

Substitute$x = cost$,$y = sint$, $z = 0, 0 < t < 2\pi$, we get

$$\int_{\theta=0}^{2\pi} (2\,cost - sint)(-sint)dt$$

$$= \int_{\theta=0}^{2\pi} (-2\,sintcost + sin^2t)dt = \int_{\theta=0}^{2\pi} (-\sin 2\,t) + (\frac{1+\cos 2\,t}{2})dt$$

$$= \left|\frac{\cos 2\,t}{2} + \frac{t}{2} - \frac{\sin 2\,t}{4}\right|_0^{2\pi} = \frac{1}{2} + \pi - \frac{1}{2} = \pi$$

Now, $\iint_S \left(\nabla \times \vec{A}\right).\hat{n}dS$

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2z \end{vmatrix}$$

$$= \left(\frac{\partial(-y^2z)}{\partial y} - \frac{\partial(-yz^2)}{\partial z}\right)\hat{\imath} - \left(\frac{\partial(-y^2z)}{\partial x} - \frac{\partial(2x-y)}{\partial z}\right)\hat{\jmath} + (\frac{\partial(-yz^2)}{\partial x} - \frac{\partial(2x-y)}{\partial y})\hat{k}$$

$$= (-2yz + 2yz)\hat{\imath} - (0-0)\hat{\jmath} + (0+1)\hat{k} = \hat{k}$$

Then

$$\iint_S \left(\nabla \times \vec{A}\right).\hat{n}dS = \iint_S \hat{k}.\hat{n}dS = \iint_R dxdy$$

(Since $\hat{k}.\hat{n}dS$)

$$= 4\int_0^1 \int_0^{\sqrt{1-x^2}} dydx = 4\int_0^1 \sqrt{1-x^2}dx$$

Let $x = \sin t \Longrightarrow dx = \cos t dt, 0 \le t \le {}^{\pi}/_{2}$. Then

$$\iint_S \left(\nabla \times \vec{A}\right).\hat{n}dS = 4\int_0^{\pi/2} \sqrt{1-\sin^2 t}\cos t\,dt$$

$$= 4\int_0^{\pi/2} \cos^2 t\,dt = \frac{4}{2}\int_0^{\pi/2} (1 + \cos 2t)dt$$

$$= \left| t + \frac{\sin 2t}{2} \right|_0^{\pi/2}$$

$$= 2\left(\pi/_2\right) = \pi$$

and stokes' theorem is verified.

Module No. 62

Another Example: Stokes' Theorem

Problem Statement

Prove

$$\oint_C d\vec{r} \times \vec{B} = \iint_S (\hat{n} \times \nabla) \times \vec{B}\, dS$$

Proof

Stokes 'theorem is

$$\oint_C \vec{A}. d\vec{r} = \iint_S (\nabla \times \vec{A}). \hat{n} dS$$

In Stokes' theorem, let $\vec{A} = \vec{B} \times \vec{C}$ where $\vec{C}$ a constant vector is. Then

$$\oint_C (\vec{B} \times \vec{C}). d\vec{r} = \iint_S (\nabla \times (\vec{B} \times \vec{C})). \hat{n} dS$$

$$\oint_C d\vec{r}. (\vec{B} \times \vec{C}) = \iint_S [(\vec{C}. \nabla)\vec{B} - \vec{C}(\nabla. \vec{B})]. \hat{n} dS$$

$$\oint_C \vec{C}. (d\vec{r} \times \vec{B}) = \iint_S [(\vec{C}. \nabla)\vec{B}]. \hat{n} dS - \iint_S [\vec{C}(\nabla. \vec{B})]. \hat{n} dS$$

$$\vec{C}. \oint_C (d\vec{r} \times \vec{B}) = \iint_S \vec{C}. [\nabla(\vec{B}. \hat{n}]dS - \iint_S \vec{C}. [\hat{n}(\nabla. \vec{B})]dS$$

$$\vec{C}. \oint_C (d\vec{r} \times \vec{B}) = \vec{C}. \iint_S [\nabla(\vec{B}. \hat{n}) - \hat{n}(\nabla. \vec{B})]dS$$

$$\vec{C}. \oint_C (d\vec{r} \times \vec{B}) = \vec{C}. \iint_S (\hat{n} \times \nabla) \times \vec{B} dS$$

Since C is an arbitrary constant vector, therefore

$$\oint_C d\vec{r} \times \vec{B} = \iint_S (\hat{n} \times \nabla) \times \vec{B} \, dS$$

Hence the result.

Module No. 63

Related Theorem: Stokes' Theorem

Theorem Statement

Prove that a necessary and sufficient condition that $\int_c \vec{A}.d\vec{r} = 0$ for every closed curve C is that $\nabla \times \vec{A} = 0$ identically.

Proof

<u>Sufficiently</u>. Suppose $\nabla \times \vec{A} = 0$. Then by the stokes' theorem

$$\oint_C \vec{A}.d\vec{r} = \iint_S (\nabla \times \vec{A}).\hat{n}dS$$

Since $\nabla \times \vec{A} = 0$, therefore $\iint_S (\nabla \times \vec{A}).\hat{n}dS = 0$

Hence

$$\oint_C \vec{A}.d\vec{r} = 0$$

<u>Necessity</u>. Suppose $\oint_C \vec{A}.d\vec{r} = 0$ around every closed path C, and assume $\nabla \times \vec{A} = 0$ at some point P. Then assuming $\nabla \times \vec{A}$ is continuous there will be a region with P as an interior point, where $\nabla \times \vec{A} = 0$. Let S be a surface contained in this region whose normal $\hat{n}$ at each point has the same direction as$\nabla \times \vec{A}$, i.e. $\nabla \times \vec{A} = \alpha\hat{n}$ where α is a positive constant. Let C be the boundary of S. Then by Stokes' theorem

$$\oint_C \vec{A}.d\vec{r} = \iint_S (\nabla \times \vec{A}).\hat{n}dS = \iint_S \alpha\hat{n}.\hat{n}dS > 0$$

which contradicts the hypothesis that $\oint_C \vec{A}.d\vec{r} = 0$ and shows that $\nabla \times \vec{A} = 0$.

Hence the theorem.

Module No. 64

Related Theorem: Stokes' Theorem

Problem Statement

Prove that

$$\oint_C \varphi d\vec{r} = \iint_S d\vec{S} \times \nabla\varphi$$

Solution

By Stokes' theorem, we have

$$\oint_C \vec{A}.d\vec{r} = \iint_S (\nabla \times \vec{A}).\hat{n}dS$$

Let $\vec{A} = \varphi\vec{C}$ where $\vec{C}$ is a constant non-zero vector, then

$$\oint_C \varphi\vec{C}.d\vec{r} = \iint_S \left(\nabla \times (\varphi\vec{C})\right).\hat{n}dS$$

or

$$\oint_C \vec{C}.\varphi d\vec{r} = \iint_S (\nabla\varphi \times \vec{C}).d\vec{S}$$

since $\nabla \times \vec{C} = \vec{0}$

$$\oint_C \vec{C}.\varphi d\vec{r} = \iint_S \nabla\varphi.\vec{C} \times d\vec{S} = \iint_S \vec{C} \times d\vec{S}.\nabla\varphi = \iint_S \vec{C}.d\vec{S} \times \nabla\varphi$$

or

$$\vec{C}.\oint_C \varphi d\vec{r} = \vec{C}.\iint_S d\vec{S} \times \nabla\varphi$$

Since $\vec{C}$ is an arbitrary constant vector, therefore

$$\oint_C \varphi d\vec{r} = \iint_S d\vec{S} \times \nabla\varphi$$

Hence the result.

Module No. 65

Further Example 2 of Stokes' Theorem

Problem Statement

If

$$\vec{A} = 2yz\hat{\imath} - (x + 3y - 2)\hat{\jmath} + (x^2 + z)\hat{k}$$

then using Stokes' theorem, evaluate

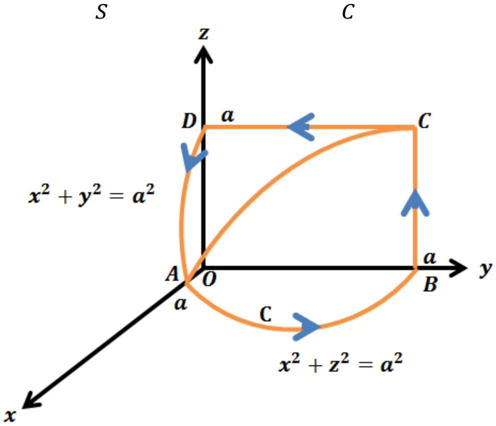
$$\iint_S (\nabla \times \vec{A}).\hat{n}dS$$

over the surface of intersection of the cylinders $x^2 + y^2 = a^2, x^2 + z^2 = a^2$ which is included in first octant.

Solution

By Stokes' theorem, we have

$$\iint_S (\nabla \times \vec{A}).\hat{n}dS = \oint_C \vec{A}.d\vec{r}$$



[1]

$$= \oint_{ABCDA} (2yz\hat{\imath} - (x + 3y - 2)\hat{\jmath} + (x^2 + z)\hat{k}).(dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k})$$

$$= \oint_{ABCDA} (2yzdx - (x + 3y - 2)dy + (x^2 + z)dz) \qquad (1)$$

For $AB, z = 0$ therefore $dz = 0$ and the integral (1) over the part of the curves becomes

$$\int_{AB} -(x + 3y - 2)dy = \int_0^a -\sqrt{a^2 - y^2} + 3y - 2)dy$$

Let $x = a\cos\theta, y = a\sin\theta,\ dy = a\cos\theta d\theta,\quad 0 \le \theta \le {}^{\pi}/_{2}$, then

$$\int_{AB} -(x+3y-2)dy = \int_{0}^{\pi/2} -(a\cos\theta + 3a\sin\theta - 2)a\cos\theta d\theta$$

$$= \int_{0}^{\pi/2} -\left[\frac{a^2}{2}(1+\cos 2\theta) + 3a^2\sin\theta\cos\theta - 2a\cos\theta\ \right]d\theta$$

$$= -\left|\frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + \frac{3}{2}a^2\sin^2\theta - 2a\sin\theta\right|_{0}^{\frac{\pi}{2}}$$

$$= -\left[\frac{a^2}{2}\left(\frac{\pi}{2}\right) + \frac{3}{2}a^2 - 2a\right] = \frac{-a^2\pi}{4} - \frac{3}{2}a^2 + 2a$$

For $BC, x = 0, y = a$ therefore $dx = dy = 0$ and integral (1) over this part of the curve becomes

$$\int_{BC} zdz = \int_{0}^{a} zdz = \frac{a^2}{2}$$

For $CD, x = 0, z = a$ therefore $dx = dz = 0$ and the integral (1) over this part of the curve becomes

$$\int_{CD} -(3y-2)dy = \int_{a}^{0} -(3y-2)dy = \frac{3a^2}{2} - 2a$$

For $DA, y = 0$ therefore $dy = 0$ and the integral (1) over this part of the curve becomes

$$\int_{DA} (x^2+2)dz = \int_{a}^{0} (a^2 - z^2 + z)dz$$

$$= \left|a^2z - \frac{z^3}{3} + \frac{z^2}{2}\right|_{a}^{0}$$

$$= -\frac{2}{3}a^3 - \frac{a^2}{2}$$

Thus from equation (1), we get

$$\iint (\nabla \times \vec{A}).\hat{n}dS = \frac{-a^2\pi}{4} - \frac{3}{2}a^2 + 2a + \frac{a^2}{2} + \frac{3a^2}{2} - 2a - \frac{2}{3}a^3 - \frac{a^2}{2}$$

$$= -\frac{a^2}{12}(3\pi + 8a)$$

required solution.

Module No. 66

Further Example 3 of Stokes' Theorem

Problem Statement

If

$$\oint_C \vec{E}.d\vec{r} = -\frac{1}{C}\frac{\partial}{\partial t}\iint_S H.dS$$

where S is any surface bounded by the curve C, show that

$$\nabla \times \vec{E} = -\frac{1}{C}\frac{\partial \vec{H}}{\partial t}.$$

Solution

As we know the Stokes' theorem

$$\oint_C \vec{A}.d\vec{r} = \iint_S (\nabla \times \vec{A}).\hat{n}dS$$

by the Stokes' theorem

$$\oint_C \vec{E}.d\vec{r} = \iint_S (\nabla \times \vec{E}).d\vec{S}$$

therefore

$$\iint_S (\nabla \times \vec{E}).d\vec{S} = -\iint_S \frac{1}{C}\frac{\partial \vec{H}}{\partial t}.d\vec{S}$$

or

$$\iint_S \left(\nabla \times \vec{E} + \frac{1}{C}\frac{\partial \vec{H}}{\partial t}\right).d\vec{S} = 0$$

This implies

$$\nabla \times \vec{E} + \frac{1}{C}\frac{\partial \vec{H}}{\partial t} = 0$$

or

$$\nabla \times \vec{E} = -\frac{1}{C}\frac{\partial \vec{H}}{\partial t}$$

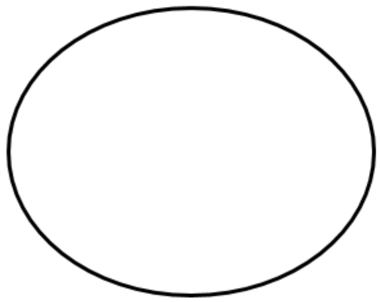
is required result.

Module No. 67

Simply and Multiply Connected Regions

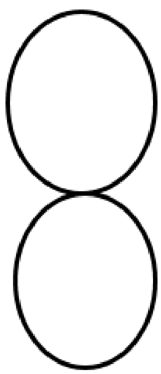
Simply Connected Region

A simple closed curve is a closed curve which does not intersects itself anywhere. For example the curve in the figure (i) is a simple closed curve



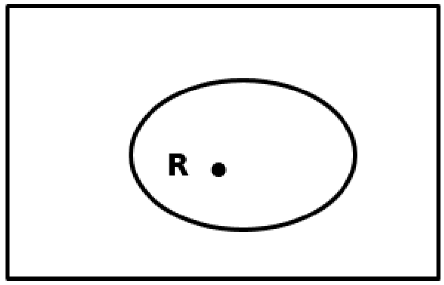
(i)

while the curve the curve in figure (ii) is not a simple closed curve.



(ii)

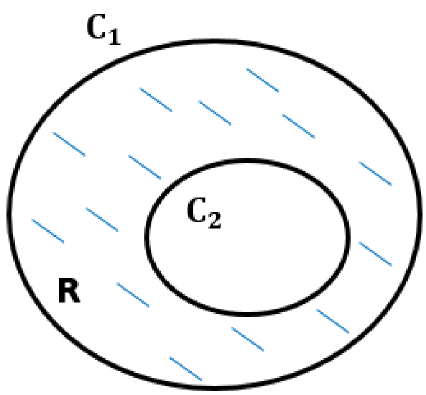
A region R which is said to be simply connected if any simple closed curve lying in R can be continuously shrunk to a point. For example, the interior of a rectangle as shown in figure (iii) is an example of simply connected region.



(iii)

Multiply Connected Regions

A region R which is not simply connected is called multiply connected. For example, the region R exterior to C_2 and interior to C_1 is not simply connected because a circle drawn within R and enclosing C_2 cannot shrunk to a point without crossing C_2 as shown in figure (iv).



(iv)

In other words, we can say that the regions which have holes are called multiply connected.

Module No. 68

Green's Theorem in the Plane

We will consider the vector function of just x and y and derive a relationship between a line integral around a closed curve and a double integral over the part of the plane enclosed by the curve.

Theorem Statement

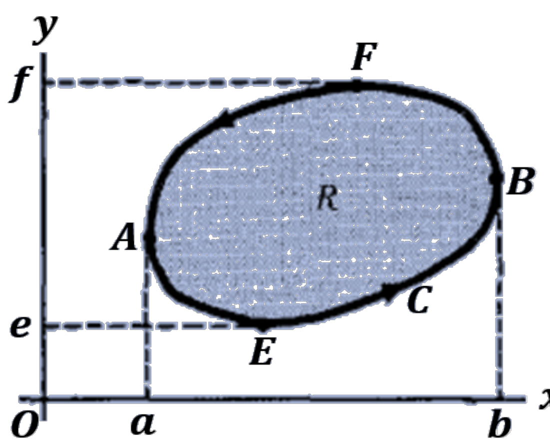
If R is simply-connected region of the xy-plane bounded by a closed curve C and if M and N are continuous functions of x and y having continuous derivatives in R, then

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

where C is described in the positive (counter-clockwise) direction.

Proof

We prove the theorem for a closed curve C which has the property that any straight line parallel to the coordinate axes cuts C in at most two points as shown in figure.



[1]

Let the equation of the curves AEB and AFB be $y = f_1(x)$ and $y = f_2(x)$ respectively. If R is the region bounded by C, we have

$$\iint_R \frac{\partial M}{\partial y} dxdy = \int_{x=a}^{b} \left[\int_{y=f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y} dy \right] dx$$

$$= \int_a^b |M(x,y)|_{y=f_1(x)}^{y=f_2(x)} dx$$

$$= \int_a^b \left[M\big(x, f_2(x)\big) - M\big(x, f_1(x)\big) \right] dx$$

$$= -\int_a^b M(x, f_1) dx - \int_a^b M(x, f_2) dx$$

$$= -\oint_C M dx$$

Then,

$$\oint_C M dx = -\iint_R \frac{\partial M}{\partial y} dxdy \qquad (1)$$

Similarly let the equation of the curves EAF and EBF be $x = g_1(y)$ and $x = g_2(y)$ respectively. Then

$$\iint_R \frac{\partial N}{\partial x} dxdy = \int_{y=e}^{f} \left[\int_{x=g_1(y)}^{g_2(y)} \frac{\partial N}{\partial x} dx \right] dy$$

$$= \int_e^f |N(x,y)|_{x=g_1(y)}^{x=g_2(y)} dy$$

$$= \int_a^b \left[N\big(x, g_2(x)\big) - N\big(x, g_1(x)\big) \right] dy$$

$$= \int_a^b N(x, g_1) dy + \int_a^b N(x, g_2) dy$$

$$= \oint_C N dy$$

Then,

$$\iint_R \frac{\partial N}{\partial x} dxdy = \oint_C Ndy \tag{2}$$

Adding equation (1) and (2), we get

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

Hence the theorem.

Module No. 69

Related Example: Green’s Theorem

Problem Statement

Verify Green's theorem in the plane for

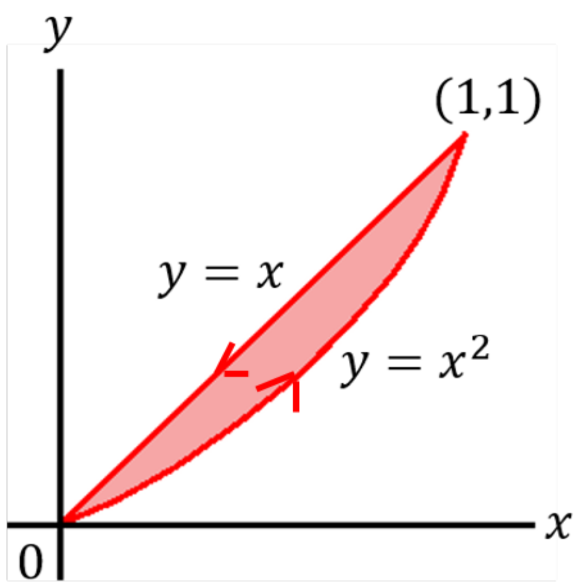
$$\oint_C (xy + y^2)\, dx + x^2\, dy$$

where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.

Solution

The plane curves $y = x$ and $y = x^2$ intersect at (0,0) and (1,1). Let C_1 be the curve $y = x^2$ and C_2 the curve $y = x$ and let the closed curve C be formed from C_1 and C_2.

The positive direction in traversing C is as shown in the adjacent diagram.



[1]

As we know the Green’s theorem

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

By comparing the given relation with Green’s theorem, we get

$M = xy + y^2$ and $N = x^2$, using these values , we must show Green’s theorem.

Now

$$\oint_C M\,dx + N\,dy = \oint_C (xy + y^2)\,dx + x^2\,dy$$

Along the curve C_1: y $= x^2, dy = 2dx$, while x varies from 0 to 1. The line Integral (1) equals to

$$\int_{C_1} M\,dx + N\,dy = \int_0^1 (3x^3 + x^4)dx$$

$$= \left|\frac{3}{4}x^4 + \frac{x^5}{5}\right|_0^1 = \frac{3}{4} + \frac{1}{5} = \frac{19}{20} \qquad (2)$$

Along the curve C_2: y $= x, dy = dx$, while x varies from 1 to 0. The line Integral (1) equals to

$$\int_{C_2} M\,dx + N\,dy = \int_1^0 2x^2dx + x^2dx = \int_1^0 3x^2dx$$

$$= |x^3|_1^0 = -1$$

Thus from equation (1) and (2), we have

$$\oint_C (xy + y^2)dx + x^2\,dy = \frac{19}{20} - 1 = -\frac{1}{20}$$

Now we will calculate the relation

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

since

$$\frac{\partial M}{\partial y} = x + 2y, \text{and } \frac{\partial N}{\partial x} = 2x$$

then

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \iint_R (2x - x - 2y)dxdy$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{x} (x - 2y)dydx = \int_0^1 (x^4 - x^3)dx$$

integrating and applying limit, we get

$$= \frac{1}{5} - \frac{1}{4} = -\frac{1}{20}$$

so the theorem is verified.

Module No. 70

Green's Theorem in the Plane in Vector Notation

First Vector Form (or tangential form) of Green's Theorem

We have Green's theorem

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy \tag{1}$$

Now

$$Mdx + Ndy = (M\hat{\imath} + N\hat{\jmath}).(dx\hat{\imath} + dy\hat{\jmath}) = \vec{A}.d\vec{r}$$

where

$\vec{A} = M\hat{\imath} + N\hat{\jmath}$ and $d\vec{r} = dx\hat{\imath} + dy\hat{\jmath}$.

Also, if $\vec{A} = M\hat{\imath} + N\hat{\jmath}$, then

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = -\frac{\partial N}{\partial z}\hat{\imath} + \frac{\partial M}{\partial z}\hat{\jmath} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k}$$

so that $(\nabla \times \vec{A}).\hat{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$

Then from equation (1) Green's theorem in the plane can be written

$$\oint_C \vec{A}.d\vec{r} = \iint (\nabla \times \vec{A}).\hat{k}dR$$

where $dR = dxdy$

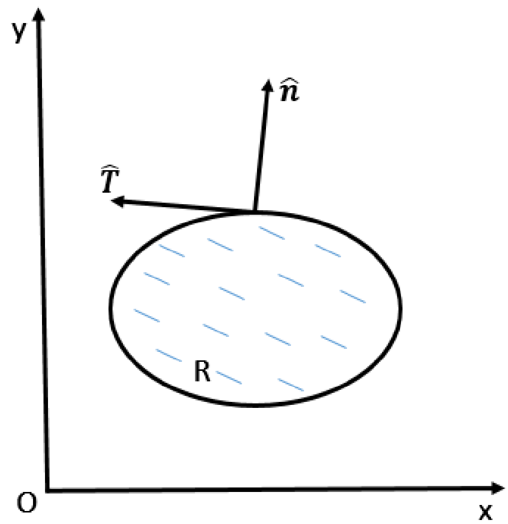
A generalization of this surface to S in space having C as boundary leads quite naturally to Stokes' theorem. This form of Green's theorem is sometimes called Stokes' theorem in the plane.

The Green's theorem in plane is a special case of Stokes theorem.

Second Vector Form (or normal form) of Green's Theorem

As we derive in first vector form of Green's theorem

$$Mdx + Ndy = \vec{A}.d\vec{r} = \vec{A}.\hat{T}ds$$



where $\hat{T} = \frac{d\vec{r}}{ds}$ =unit tangent vector to C as shown in figure.

If $\hat{n}$ is the outward drawn unit normal to C, then $\hat{T} = \hat{k} \times \hat{n}$. so that

$$Mdx + Ndy = \vec{A}.\hat{T}ds = \vec{A}.\left(\hat{k} \times \hat{n}\right)ds = \left(\vec{A} \times \hat{k}\right).\hat{n}ds$$

Since $\vec{A} = M\hat{\imath} + N\hat{\jmath}$, therefore

$$\vec{B} = \vec{A} \times \hat{k} = (M\hat{\imath} + N\hat{\jmath}) \times \hat{k} = N\hat{\imath} - M\hat{\jmath}$$

and

$$\nabla.\vec{B} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

then the equation (1) becomes

$$\oint_C \vec{B}.\hat{n}\,ds = \iint_S \nabla.\vec{B}\,dR$$

where $dR = dxdy$

these are the required vector notations of Green's theorem.

Module No. 71

Green's Theorem in the Plane as Special case of Stokes' Theorem

Green’s theorem can be expressed in the plane vector notations which are also named the tangential form or normal forms of Green’s theorem.

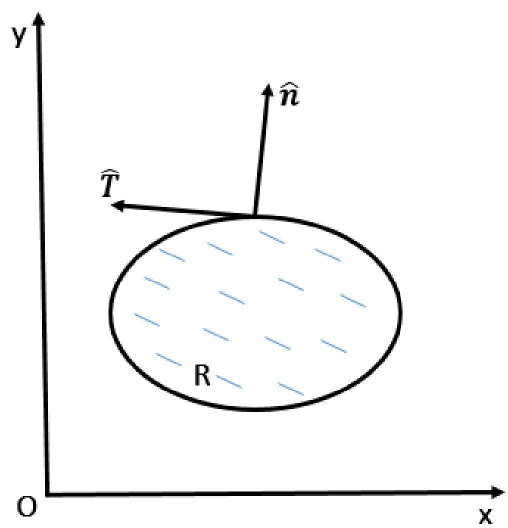
The tangential form of Green’s theorem is also called the first vector form of Green’s theorem. This generalize form of Green’s theorem in plane also called Stokes’ theorem in the plane. Thus we can say that Green’ theorem is a special case of Stokes’ theorem when applied to a region in the xy-plane.

Module No. 72

Gauss' Divergence Theorem as Generalization of Green's Theorem

As we derive in first vector form of Green's theorem

$$Mdx + Ndy = \vec{A}.d\vec{r} = \vec{A}.\hat{T}ds$$



where $\hat{T} = \frac{d\vec{r}}{ds}$ =unit tangent vector to C as shown in figure.

If $\hat{n}$ is the outward drawn unit normal to C, then $\hat{T} = \hat{k} \times \hat{n}$. so that

$$Mdx + Ndy = \vec{A}.\hat{T}ds = \vec{A}.\left(\hat{k} \times \hat{n}\right)ds = \left(\vec{A} \times \hat{k}\right).\hat{n}ds$$

Since $\vec{A} = M\hat{\imath} + N\hat{\jmath}$, therefore

$$\vec{B} = \vec{A} \times \hat{k} = (M\hat{\imath} + N\hat{\jmath}) \times \hat{k} = N\hat{\imath} - M\hat{\jmath}$$

and

$$\nabla.\vec{B} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

then the equation (1) becomes

$$\oint_C \vec{B}.\hat{n}\, ds = \iint_S \nabla.\vec{B}\, dR$$

where $dR = dxdy$

these are the required vector notations of Green’s theorem.

The generalization of Green’s theorem as Gauss’s divergence theorem is also called the second vector form (normal form) of Green’s theorem.

Generalization of this to the case where the differential arc length ds of a closed curve C is replaced by the differential of surface area dS of a closed surface S, and the corresponding plane region R enclosed by C is replaced by the volume V enclosed by S, leads to Gauss' divergence theorem or Green's theorem in space.

$$\iint_S \vec{B}.\hat{n}dS = \iiint_R \nabla.\vec{B}dV$$

Module No. 73

Green's First Identity

Theorem Statement

If φ and ψ are scalar point functions with continuous second order derivatives in a region R bounded by a closed surface S, then

$$\iiint_R [\varphi\nabla^2\psi + (\nabla\varphi)(\nabla\psi)]\, dV = \iint_S (\varphi\nabla\psi)\,.\, d\vec{S}$$

Proof

Since divergence theorem is

$$\iint_S \vec{A}.\hat{n}dS = \iiint_R \nabla.\vec{A}dV$$

Now substitute $\vec{A} = \varphi\nabla\psi$ in the divergence theorem, we obtain

$$\iiint_R \nabla.(\varphi\nabla\psi)dV = \iint_S (\varphi\nabla\psi).\hat{n}dS = \iint_S (\varphi\nabla\psi)\, d\vec{S} \qquad (1)$$

But

$$\nabla.(\varphi\nabla\psi) = (\nabla\varphi).(\nabla\psi) + \varphi(\nabla.\nabla\psi) = \varphi\nabla^2\psi + (\nabla\varphi)(\nabla\psi)$$

thus the equation (1) becomes

$$\iiint_R [\varphi\nabla^2\psi + (\nabla\varphi)(\nabla\psi)]\, dV = \iint_S (\varphi\nabla\psi)\, d\vec{S}$$

Hence the theorem.

Alternative Forms of Green's first Identity

We know that

$$\nabla\psi.\hat{n} = \frac{\partial\psi}{\partial n} \; and \; \nabla\varphi.\hat{n} = \frac{\partial\varphi}{\partial n}$$

Thus

$$\nabla\psi . d\vec{S} = \nabla\psi . \hat{n}dS = \frac{\partial\psi}{\partial n}dS$$

and

$$\nabla\varphi . d\vec{S} = \nabla\varphi . \hat{n}dS = \frac{\partial\varphi}{\partial n}dS$$

Hence the Green's first Identity can be written as

$$\iiint_R [\varphi\nabla^2\psi + (\nabla\varphi)(\nabla\psi)]\, dV = \iint_S \varphi\frac{\partial\psi}{\partial n}dS$$

Module No. 74

Green's Second Identity

If φ and ψ are scalar point functions with continuous second order derivatives in a region R bounded by a closed surface S, then

$$\iiint_R [\varphi\nabla^2\psi - \psi\nabla^2\varphi]\, dV = \iint_S (\varphi\nabla\psi - \psi\nabla\varphi)\, .\, d\vec{S}$$

Proof

We have Green's first identity

$$\iiint_R [\varphi\nabla^2\psi + (\nabla\varphi)(\nabla\psi)]\, dV = \iint_S (\varphi\nabla\psi)\, .\, d\vec{S} \qquad (1)$$

Interchanging φ and ψ in equation (1), we obtain

$$\iiint_R [\psi\nabla^2\varphi + (\nabla\psi)(\nabla\varphi)]\, dV = \iint_S (\psi\nabla\varphi)\, .\, d\vec{S} \qquad (2)$$

Subtracting equation (2) from (1), we have

$$\iiint_R [\psi\nabla^2\varphi + (\nabla\psi)(\nabla\varphi)] - [\varphi\nabla^2\psi + (\nabla\varphi)(\nabla\psi)]\, dV = \iint_S (\psi\nabla\varphi) - (\varphi\nabla\psi)\, .\, d\vec{S}$$

$$\iiint_R [\varphi\nabla^2\psi - \psi\nabla^2\varphi]\, dV = \iint_S (\varphi\nabla\psi - \psi\nabla\varphi)\, .\, d\vec{S}$$

which is called Green's second identity or symmetrical theorem.

Alternative Forms of Green's Second Identity

We know that

$$\nabla\psi.\hat{n} = \frac{\partial\psi}{\partial n} \; and \; \nabla\varphi.\hat{n} = \frac{\partial\varphi}{\partial n}$$

Thus

$$\nabla\psi.d\vec{S} = \nabla\psi.\hat{n}dS = \frac{\partial\psi}{\partial n}dS$$

and

$$\nabla\varphi . d\vec{S} = \nabla\varphi . \hat{n} dS = \frac{\partial\varphi}{\partial n} dS$$

Hence the Green's Second Identity can be written as

$$\iiint_R [\varphi\nabla^2\psi - \psi\nabla^2\varphi]\, dV = \iint_S (\varphi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\varphi}{\partial n})\, dS$$

Module No. 75

Related Example: Green's Theorem

Problem Statement

Evaluate

$$\int_{(0,0)}^{(2,1)} (10x^4 - 2xy^3)dx - 3x^2y^2dy$$

along the path $x^4 - 6xy^3 = 4y^2$.

Solution

A direct evaluation is difficult. By comparing it with Green's Theorem, we get

$$\oint_C Mdx + Ndy = \int_{(0,0)}^{(2,1)} (10x^4 - 2xy^3)dx - 3x^2y^2dy$$

here $M = 10x^4 - 2xy^3$ and $N = -3x^2y^2$ and

$$\frac{\partial M}{\partial y} = -6x^2y = \frac{\partial N}{\partial x}$$

it follows that the integral is independent of the path. Then we can use any path, for example the path consisting of straight line segments from (0,0) to (2,0) and then from (2,0) to (2,1).

Along the straight line path from (0,0) to (2,0), $y = 0, dy = 0$ and the integral equals

$$\int_{x=0}^{2} 10x^4dx = \frac{10}{5}x^5 = 2(32) = 64$$

Along the straight line path from (2,0)to (2,1), $x = 2, dx = 0$ and the integral equals

$$\int_{y=0}^{1} -12y^2dy = -12\left(\frac{y^3}{3}\right) = -4(1) = -4$$

Then the value of the line integral

$$\int_{(0,0)}^{(2,1)} (10x^4 - 2xy^3)dx - 3x^2y^2dy = 64 - 4 = 60$$

is the required solution.

Module No. 76

Selected Problem 1: Green's Theorem

Problem Statement

Prove that

$$\oint Mdx + Ndy = 0$$

$Mdx + Ndy = 0$ around every closed curve C in a simply-connected region if and only

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

everywhere in the region.

Proof

Assume that M and N are continuous and have continuous partial derivatives everywhere in the region R bounded by C, so that Green's theorem is applicable.

Then

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

<u>Sufficient:</u> If

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

in R, then Clearly

$$\oint Mdx + Ndy = 0$$

<u>Necessity:</u> suppose

$$\oint Mdx + Ndy = 0$$

for all curves C. If $\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}>0$ at a point P, then from the continuity of the derivatives it follows that $\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}>0$ in some region A surrounding P. If Γ is the boundary of A then

$$\oint_C Mdx+Ndy=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy>0$$

which contradicts the assumption that the line integral is zero around every closed curve. Similarly the assumption $\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}<0$ leads to a contradiction. Thus $\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}=0$ at all points on R.

Module No. 77

Selected Problem 2: Green's Theorem

Problem Statement

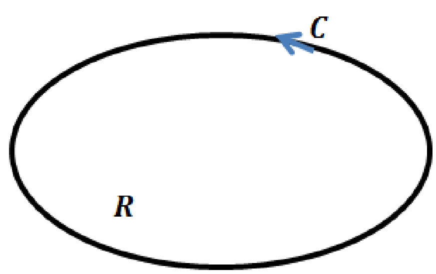
Show that the area bounded by a simple closed curve C is given by

$$\frac{1}{2}\oint_C xdy - ydx$$

Proof

Since Green's theorem is

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$



Put $M = -y$ and $N = x$ in Green's theorem, we get

$$\oint_C xdt - ydx = \iint_R \left(\frac{\partial(x)}{\partial x} - \frac{\partial(-y)}{\partial y}\right) dxdy = 2\iint_R dxdy = 2\text{A}$$

$$\oint_C xdt - ydx = 2\text{A}$$

or

$$\text{Area} = \text{A} = \frac{1}{2}\oint_C xdy - ydx$$

Hence the result.

Now we will illustrate this formula through an example

Example Statement

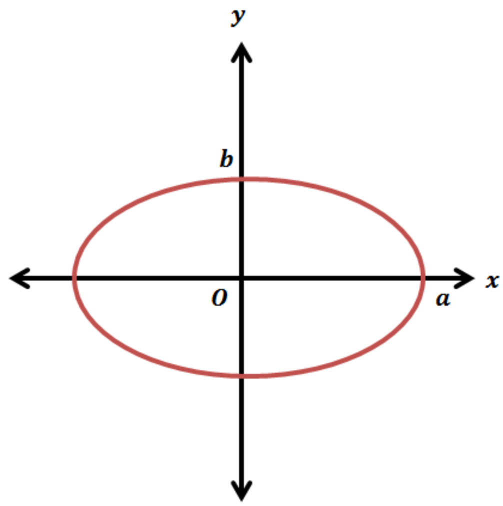
Find the area of the ellipse $x = a\,cos\theta, \mathrm{y} = b\,sin\theta$

Solution

By making use of above result,

$$\text{Area} = \text{A} = \frac{1}{2}\oint_C xdy - ydx$$

$$= \frac{1}{2}\int_0^{2\pi} (a\,cos\theta)(b\,cos\theta)\,d\theta - (b\,sin\theta)(-a\,sin\theta)d\theta$$



$$= \frac{1}{2}\int_0^{2\pi} ab\,cos^2\theta\,d\theta + absin^2\theta d\theta = \frac{1}{2}\int_0^{2\pi} ab\,(cos^2\theta + sin^2\theta)d\theta$$

$$= ab.\frac{1}{2}\int_0^{2\pi} d\theta = ab\pi$$

is the required area.

Module No. 78

Selected Problem 3: Green’s Theorem

Problem Statement

Evaluate

$$\oint_C (y - \sin x)dx + \cos x\, dy$$

where C is the triangle of the adjoining figure:

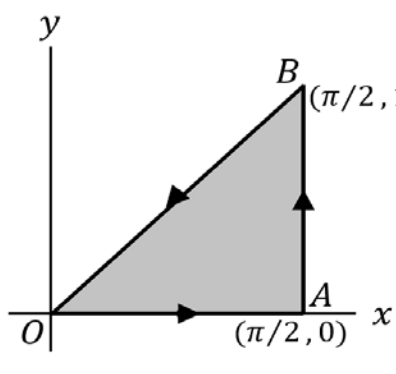
i. Directly,

ii. By using Green's theorem in the plane.

Solution

i. Along OA, $y = 0, dy = 0$ and the integral equals

$$\int_0^{\pi/2} (0 - \sin x)dx + \cos x\,(0) = \int_0^{\pi/2} |{-\sin x}dx = \cos x\,|_0^{\pi/2} = -1$$



Along AB, $x = {}^{\pi}/_{2}$, $dx = 0$ and the integral equals to

$$\int_0^1 (y-1)(0) + 0dy = 0$$

Along BO, y $= \frac{2x}{\pi}$, $dy = \frac{2}{\pi}dx$ and the integral equals

$$\int_{\pi/2}^{0} (\frac{2x}{\pi} - \sin x)dx + \frac{2}{\pi}\cos x\, dx = \left|\frac{x^2}{\pi} + \cos x + \frac{2}{\pi}\sin x\right|_{x=\pi/2}^{0}$$

$$= 1 - \frac{\pi}{4} - \frac{2}{\pi}$$

Then the integral along C $= -1 + 0 + 1 - \frac{\pi}{4} - \frac{2}{\pi} = -\frac{\pi}{4} - \frac{2}{\pi}$

Hence

$$\oint_C (y - \sin x)dx + \cos x\, dy = -\frac{\pi}{4} - \frac{2}{\pi}$$

ii. Since the green's theorem is

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy \qquad (1)$$

by comparing with the given integral, we get

$$\oint_C Mdx + Ndy = \oint_C (y - \sin x)dx + \cos x\, dy$$

$M = y - \sin x,\qquad N = \cos x\, , \frac{\partial M}{\partial y} = 1\ \frac{\partial N}{\partial x} = -\sin x$

then the equation (1) becomes

$$= \oint_C (y - \sin x)dx + \cos x\, dy = \iint_R (-\sin x - 1)dxdy$$

$$= \int_{x=0}^{\pi/2} \left[\int_{y=0}^{\frac{2x}{\pi}} (-\sin x - 1)dy\right] dx$$

$$= \int_{x=0}^{\pi/2} |-\mathrm{y}\sin x - y|_0^{\frac{2x}{\pi}}\, dx$$

$$\int_{x=0}^{\pi/2} \left(-\frac{2x}{\pi}\sin x - \frac{2x}{\pi}\right) dx$$

$$\left| -\frac{2}{\pi}(-x\cos x + \sin x) - \frac{\pi^2}{2} \right|_{x=0}^{x=\pi/2}$$

$$= -\frac{\pi}{4} - \frac{2}{\pi}$$

in agreement with part (i).