Lecture Handouts

on

VECTORS AND CLASSICAL MECHANICS (MTH-622)

Virtual University of Pakistan
Pakistan

About the Handouts

The following books have been mainly followed to prepare the slides and handouts:

1. Spiegel, M.R., *Theory and Problems of Vector Analysis: And an Introduction to Tensor Analysis*. 1959: McGraw-Hill.
2. Spiegel, M.S., *Theory and problems of theoretical mechanics*. 1967: Schaum.
3. Taylor, J.R., *Classical Mechanics*. 2005: University Science Books.
4. DiBenedetto, E., *Classical Mechanics: Theory and Mathematical Modeling*. 2010: Birkhäuser Boston.
5. Fowles, G.R. and G.L. Cassiday, *Analytical Mechanics*. 2005: Thomson Brooks/Cole.

The first two books were considered as main text books. Therefore the students are advised to read the first two books in addition to these handouts. In addition to the above mentioned books, some other reference book and material was used to get these handouts prepared.

Contents

Module No. 101

Example of Non- Conservative Field

Problem:

Show that the force field given by

$$\vec{F} = x^2yz\hat{\imath} - xyz^2\hat{\jmath} + 2xz\hat{k}$$

is non-conservative.

Solution:

We have

$$\vec{F} = x^2yz\hat{\imath} - xyz^2\hat{\jmath} + 2xz\hat{k}$$

To verify whether the force field F is conservative or not, we will check whether $\nabla \times \vec{F} = 0$ or not.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2yz & -xyz^2 & 2xz \end{vmatrix}$$

$$= \hat{\imath}\left[\frac{\partial}{\partial y}(2xz) - \frac{\partial}{\partial z}(-xyz^2)\right] - \hat{\jmath}\left[\frac{\partial}{\partial z}(x^2yz) - \frac{\partial}{\partial x}(2xz)\right] + \hat{k}\left[\frac{\partial}{\partial x}(-xyz^2) - \frac{\partial}{\partial y}(x^2yz)\right]$$

$$= (2xyz)\hat{\imath} - (x^2y - 2z)\hat{\jmath} + (yz^2 + x^2z)\hat{k}$$

$$\neq 0$$

As we obtain $\nabla \times \vec{F} \neq 0$

We conclude that $\vec{F}$ is non-conservative.

Module No. 102

Introduction to Simple Harmonic Motion and Oscillator

Simple Harmonic Motion (SHM) is a particular type of oscillation and periodic motion in which restoring force of an object is directly proportional to the displacement of the object acting in opposite direction of displacement.

Mathematically, the restoring force F is given by

$$F_R = -kx$$

where the subscript R represents the restoring force and k is the constant of proportionality often called the spring constant or modulus of elasticity.

In Newtonian mechanics, by Newton's second law we have eq. of S.H.M

$$F = ma = m\frac{d^2x}{dt^2} = -kx$$

or

$$m\ddot{x} + kx = 0$$

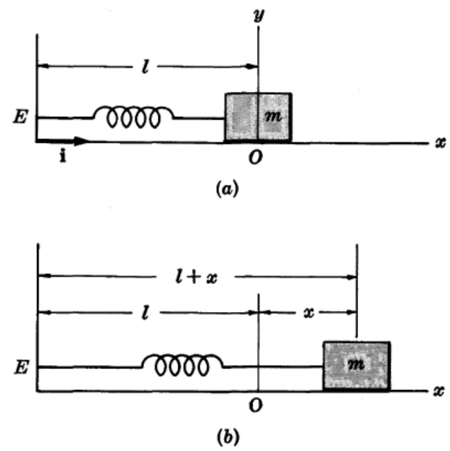
This vibrating system is called a simple harmonic oscillator or linear harmonic oscillator.

This type of motion is often called simple harmonic motion.

The following physical systems are some examples of simple harmonic oscillator

Mass on a spring

An object of mass *m* linked to a spring of spring constant *k* represents the simple harmonic motion in closed region.



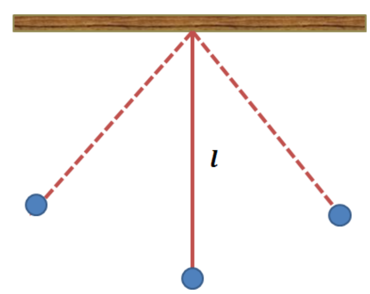
The equation representing the period for the mass attached on a spring

$$T = 2\pi\sqrt{\frac{m}{k}}$$

The above equation expresses that the time period of oscillation is independent of amplitude as well as the acceleration.

Simple pendulum

The movement of the mass attached to a simple pendulum is considered as simple harmonic motion.



The time period of a mass m attached to a pendulum of length *l* with gravitational acceleration g is given by

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Module No. 103

Amplitude, Time period, Frequency and Energy of S.H.M

Amplitude

Amplitude is defined as the maximum distance covered by the oscillating body in one oscillation or length of a wave measured from its mean position.

The amplitude of a pendulum is one-half the distance that the mass covered in moving from one terminal to the other. The vibrating sources generate waves, whose amplitude is proportional to the amplitude of the vibrating source.

Time Period

Time period is minimum time required by a oscillation system to complete its one cycle of oscillation of the specific system.

It is denoted by T and measured in seconds.

Frequency

The frequency (f) of an oscillatory system is the number of oscillations pass through a specific point in one second.

It is measure in hertz (Hz).

The frequency of S.H.M can be calculate by using the following relation

$$f = \frac{1}{T}$$

Energy of S.H.M

If T is the kinetic energy, V the potential energy and $E = T + V$ the total energy of a simple harmonic oscillator, then we have

$$K.E = T = \frac{1}{2}mv^2$$

and

$$V = \frac{1}{2}kx^2$$

Then the total energy of S.H.M will be

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

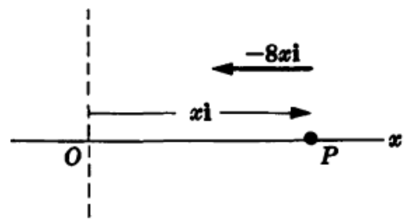
Module No. 104

Example of S.H.M

Problem Statement

A particle of mass 4 moves along y axis attracted towards the origin by a force whose magnitude is numerically equal to $6y$. If the particle is initially at rest at $y = 10$,

i. The differential equation and initial conditions describing the motion,
ii. The position of the particle at any time,
iii. The speed and velocity of the particle at any time,
iv. The amplitude, time period and frequency of the vibration.



Solution:

i. Let $r = y$ be the position vector of the particle. The acceleration of the particle is

$$\frac{d^2r}{dt^2} = \frac{d^2y}{dt^2}$$

The net force acting on the particle is $-6y$. Then by using the Newton's second law,

$$-6y = 4\frac{d^2y}{dt^2}$$

or $\qquad \frac{d^2y}{dt^2} + \frac{3}{2}y = 0$

which is the required differential equation. The required initial conditions (I. Cs) are

$y = 10,\ dy/dt = 0\ at\ t = 0$

ii. Since the general solution the diff. eq. is

When $t = 0, x = 20\ so\ that\ A = 20$. Thus

$$y = A\cos\sqrt{3/2}t + B\sin\sqrt{3/2}\,t \qquad (1)$$

Using I. Cs

At $t = 0,\ y = 10$. Thus $A = 10$. So,

$$y = 10\cos\sqrt{3/2}t + \text{B}\sin\sqrt{3/2}\,t \qquad (2)$$

$\frac{dy}{dt} = -10\sqrt{3/2}\sin\sqrt{3/2}\,t + \sqrt{3/2}\,\mathrm{B}\cos\sqrt{3/2}t$

(3)

Since at $t = 0, \Rightarrow \frac{dy}{dt} = 0$

Thus, we get $B = 0.$ Hence (2) becomes

$y = 10\cos\sqrt{3/2}t$ (4)

This gives the position at any time.

iii. Speed at any time?

From (6) $\frac{dy}{dt} = -10\sqrt{3/2}\sin\sqrt{3/2}t$

This gives the speed at any time.

iv. Amplitude, T, F ?

General form for the position of a particle moving to and fro is

$$y(t) = \mathrm{A}\cos(2\pi t/\mathrm{T})$$

Thus, $y = 10\cos\sqrt{3/2}t$

$$\text{Amplitude} = 10$$

Period $= \mathrm{T} = 2\sqrt{\frac{2}{3}}\pi$

$$\text{Frequency} = 1/T$$

Module No. 105

Example of Energy of S.H.M

ENERGY OF A SIMPLE HARMONIC OSCILLATOR

Example:

Find the total energy of the force $\vec{F} = -3xi$ acting on a simple harmonic oscillator, where i represents the direction.

Solution:

Since the total energy of SHM

$$E = T + V$$

By Newton's second law,

$$F = ma$$

therefore

$$F = m\frac{dv}{dt} = -3x$$

$$\frac{dv}{dt} = -\frac{3}{m}x$$

Integrating with respect to t, we have

$$v = -\frac{3}{2m}x^2 + c$$

c can be calculated if the initial condition is given. Assuming $v = 0$ initially, which gives $c = 0$.

Thus $\qquad v = -\frac{3}{2m}x^2$

So,

$$T = \frac{9}{8m}x^4$$

Now, as the potential energy is given by V where $\vec{F} = -\nabla V$

or $\qquad F = -3xi = -(\frac{\partial V}{\partial x}i + \frac{\partial V}{\partial y}j + \frac{\partial V}{\partial z}k)$

Then

$$\frac{\partial V}{\partial x} = 3x\,, \ \frac{\partial V}{\partial y} = 0, \ \frac{\partial V}{\partial z} = 0$$

By integrating, we obtain

$$V = \frac{3}{2}x^2 + c_1$$

Assuming $V = 0$, corresponding to $x = 0$, we get $c_1 = 0$.

So the potential energy is $\quad V = \frac{1}{2}kx^2$

Hence

$$E = \frac{9}{8m}x^4 + \frac{1}{2}kx^2$$

Module No. 106

Damped Harmonic Oscillator

The forces acting on a harmonic oscillator are called damping forces which tend to decrease the amplitude of the successive oscillations or simply force apposing the motion. Since the damping force is proportional to the velocity. Thus, mathematically,

$$F_d = -bv$$

$$= -b\dot{x}$$

where d represents the damping force and b is the damping coefficient.

The negative sign shows that that direction of F_d is opposite to the velocity v.

As we know that for the restoring force:

$$m\ddot{x} + kx = 0$$

Adding the restoring force with the damping force, the equation of motion of the damped harmonic oscillator will be,

$$m\ddot{x} + kx = -b\dot{x}$$

or

$$m\ddot{x} + b\dot{x} + kx = 0$$

Since the mass is not equal to zero, therefore $\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$

let

$$\frac{b}{m} = 2\zeta,\ \sqrt{\frac{k}{m}} = \omega_0$$

then the equation can be written as

$$\ddot{x} + 2\zeta\dot{x} + \omega_0^2 x = 0$$

Module No. 107

Example of Damped Harmonic Oscillator

Problem Statement

A particle of mass 4 moves along y axis attracted towards the origin by a force whose magnitude is numerically equal to $6y$. If the particle is initially at rest at y = 10. Consider that the particle has also a damping force whose magnitude is numerically equal to 6 times the instantaneous speed.

Find

i. Position of the particle
ii. Velocity of the particle at any time

Solution

i. Since we have by Newton's second law

$$F = ma = 4\ddot{y}$$

According to the given information, the damping force is $-6\dot{y}$.
So the net force is

$$-6y - 6\dot{y}$$

Hence,

$$4\ddot{y} = -6y - 6\dot{y}$$

or

$$\ddot{y} + \frac{2}{3}\dot{y} + \frac{2}{3}y = 0$$

The solution of the above equation is

$$y = e^{-\sqrt{2/3}t}(A + Bt)$$

and

$$\dot{y} = Be^{-\sqrt{2/3}t} - \sqrt{2/3}\,(A + Bt)e^{-\sqrt{2/3}t}$$

To find the values of constants, we use I. Cs
At $t = 0$, $y = 10$ and $dy/dt = 0$; thus,

$$A = 10$$

and $$0 = B(1) - \sqrt{2/3}\,(10)$$

$$B = 10\sqrt{2/3}$$

and the solution gives

$$y = e^{-\sqrt{2/3}t}(A + Bt)$$
$$= 10e^{-\sqrt{2/3}t}(1 + \sqrt{2/3}\,t)$$

the position at any time t.

Module No. 108

Euler's Theorem – Derivation

The following theorem, called Euler's theorem, is fundamental in the motion of rigid bodies.

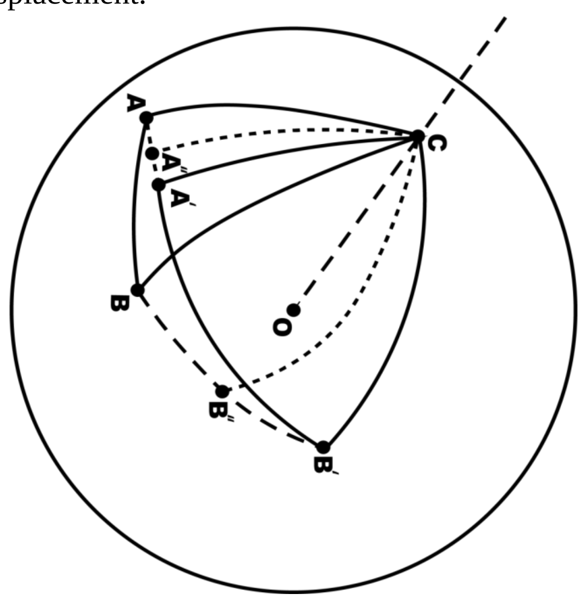
Theorem Statement

"A rotation of a rigid body about a fixed point of the body is equivalent to a rotation about a line which passes through the point."

The line referred to is called the instantaneous axis of rotation. Rotations can be considered as finite or infinitesimal. Finite rotations cannot be represented by vectors since the commutative law fails. However, infinitesimal rotations can be represented by vectors.

Proof:

Let O be the fixed point in the body, which we take as a sphere S. Further, we take O at the center of the sphere. Let A, B be two distinct points on the sphere. As the body moves, the point O (on the axis) remains foxed and A and B suffer displacement.



Let A' and B' bet the new locations of the points A and B after an infinitesimal time interval δt respectively.We join (A, B) and (A', B') by great circular areas.
Also we join (A, A') and (B, B') by mean of great circular arcs. Let A" and B" draw axes at right angles, which meat at the point C on the sphere.
We join C with A, B, A', B' by means of great circular arcs.
Consider the spherical triangles $\Delta CA'A"$ and $\Delta CAA"$. Obviously

i. $m < CA''A \cong CA''A'$
ii. $AA''=A''A'$
iii. CA'' is common to triangle

Therefore $\Delta CAA'' \cong \Delta CA'A''$ and it follows that $CA = CA'$
Similarly for the triangle ΔCAB and $\Delta CA'B'$, we have

$$CB = CB', \quad CA' = CA \ \ and \ \ AB = A'B'$$

The last relation is due to the fact that by the definition of rigid body, the distance between a pair of particles remains unchanged. Hence

$$\Delta CAB \cong \Delta CA'B'$$

The portion of rigid body lying in ΔCAB has moved to $\Delta CA'B'$.
In this process the point O and C have remained fixed, Although the later was at rest only instantaneously. Therefore the body has under gone a rotation about the axis OC.
Hence the theorem.

Module No. 109

Chasles' Theorem

Theorem Statement:

Chasle's theorem states that the most general rigid body displacement can be produced by a translation along a line (called its screw axis) followed (or preceded) by a rotation about that line.

Explanation:

- A rigid body has six degrees of freedom.
- By Euler's theorem, three of these are associated with pure rotation.
- The remaining three must be associated with translation.
- To describe the general motion of a rigid body, think of the general motion as translation of a fixed point O in the body to a point O' followed by the rotation about an axis through O'.

Module No. 110

Kinematics of a System of Particles (Space, time & matter)

Kinematics is the branch of mechanics deals with the moving objects without reference to the forces which cause the motion.
In other words we can say those kinematics are the features or properties of motion of concerned with system of particles (rigid bodies).
Here some features of rigid body motion are

- Displacement
- Position
- Velocity
- Linear Velocity & Angular Velocity
- Linear Acceleration & Angular Acceleration
- Motion of a Rigid Body (Translation & Rotation)

From everyday experience, we all have some idea as to the meaning of each of the following terms or concepts. However, we would certainly find it difficult to formulate completely satisfactory definitions. We take them as undefined concepts.

I. **Space.** This is closely related to the concepts of point, position,' direction and displacement. Measurement in space involves the concepts of length or distance, with which we assume familiarity. Units of length are feet, meters, miles, etc.

II. **Time**. This concept is derived from our experience of having one event taking place after, before or simultaneous with another event. Measurement of time is achieved, for example, by use of clocks. Units of time are seconds, hours, years, etc.

III. **Matter.** Physical objects are composed of "small bits of matter" such as atoms and molecules. From this we arrive at the concept of a material object called a particle which can be considered as occupying a point in space and perhaps moving as time goes by. A measure of the "quantity of matter" associated with a particle is called its mass. Units of mass are grams, kilograms, etc. Unless otherwise stated we shall assume that the mass of a particle does not change with time.

Module No. 111

The Concept of Rectilinear Motion of Particles, Uniform Rectilinear Motion, Uniformly Accelerated Rectilinear Motion

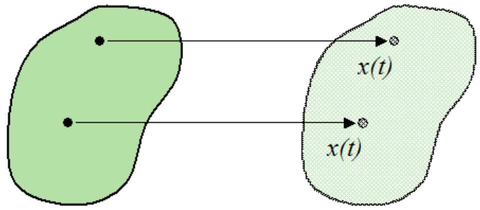
When a moving particle remains on a single straight line, the motion is said to be rectilinear. In this case, without loss of generality we can choose the x-axis as the line of motion.
The general equation of motion is then

$$F = ma \Longrightarrow F(x, \dot{x}, \ddot{x}) = m\ddot{x}$$

Rectilinear motion for a particle:

Rectilinear motion of a body is defined by considering the two point of a body covered the same distance in the parallel direction. The figures below illustrate rectilinear motion for a particle and body.

Rectilinear motion for a body:



In the above figures, $x(t)$ represents the position of the particles along the direction of motion, as a function of time t.
An example of linear motion is an athlete running g along a straight track.
The rectilinear motion can be of two types:

i. Uniform rectilinear motion
ii. Non uniform rectilinear motion

Uniform Rectilinear Motion:

Uniform rectilinear motion is a type of motion in which the body moves with uniform velocity or zero acceleration.

In contrast, Non uniform rectilinear motion is such type of motion with variable velocity or non-zero acceleration.

Uniformly accelerated rectilinear motion:

Uniformly accelerated rectilinear motion is a special case of non-uniform rectilinear motion along a line is that which arises when an object is subjected to constant acceleration. This kind of motion is called uniformly accelerated motion.

Uniformly accelerated motion is a type of motion in which the velocity of an object changes by an equal amount in every equal intervals of time.

An example of uniformly accelerated body is freely falling object in which the amount of gravitational acceleration remains same.

$$F = mg$$

Module No. 112

The Concept of Curvilinear Motion of Particles

The motion of a particle moving in a curved path is called curvilinear motion. Example: A stone thrown into the air at an angle.
Curvilinear motion describes the motion of a moving particle that conforms to a known or fixed curve. The study of such motion involves the use of two co-ordinate systems, the first being planar motion and the latter being cylindrical motion.

Tangential and normal unit vectors are usually denoted by $\overrightarrow{e_t}$ and $\overrightarrow{e_n}$ respectively.

Velocity of Curvilinear motion

If the tangential and normal unit vectors are $\overrightarrow{e_t}$ and $\overrightarrow{e_n}$ respectively, then the velocity will be

$$\vec{v} = \frac{ds}{dt}\overrightarrow{e_t}$$

You have already learnt that

$$v = \mathrm{vT}$$

Acceleration of Curvilinear motion

If the tangential and normal unit vectors are $\overrightarrow{e_t}$ and $\overrightarrow{e_n}$ respectively, then the acceleration will be

$$\vec{a} = \frac{d^2s}{dt^2}\overrightarrow{e_t} + \frac{\left({ds}/{dt}\right)^2}{\rho}\overrightarrow{e_n}$$

You have already learnt that

$$a = \mathrm{T}\frac{d\mathrm{v}}{dt} + \frac{\mathrm{v}^2}{\mathrm{r}}\mathrm{N}$$

Example

- A stone thrown into the air at an angle.
- A car driving along a curved road.
- Throwing paper airplanes or paper darts is an example of curvilinear motion.

Module No. 113

Example Related to Curvilinear Coordinates

Example 1

Problem Statement

Prove that a cylindrical coordinate system is orthogonal.

Solution

The position vector of any point in cylindrical coordinates is

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

We studied that the in cylindrical coordinates

$$x = \rho\cos\varphi\,, \qquad y = \rho\sin\varphi, \qquad z = z$$

this implies

$$\vec{r} = \rho\cos\varphi\,\hat{\imath} + \rho\sin\varphi\,\hat{\jmath} + z\hat{k}$$

The tangent vectors to the ρ, φ and r curves are given respectively by $\frac{\partial r}{\partial\rho}$,$\frac{\partial r}{\partial\varphi}$,$\frac{\partial r}{\partial z}$ and where

$$\frac{\partial r}{\partial\rho} = \cos\varphi\,\hat{\imath} + \sin\varphi\,\hat{\jmath}$$

$$\frac{\partial r}{\partial\varphi} = -\rho\sin\varphi\,\hat{\imath} + \rho\cos\varphi\,\hat{\jmath}$$

$$\frac{\partial r}{\partial z} = \hat{k}$$

The unit vectors in these directions are

$$e_1 = e_\rho = \frac{\partial r/\partial\rho}{|\partial r/\partial\rho|} = \frac{\cos\varphi\,\hat{\imath} + \sin\varphi\,\hat{\jmath}}{|\cos\varphi\,\hat{\imath} + \sin\varphi\,\hat{\jmath}|}$$

$$= \frac{\cos\varphi\,\hat{\imath} + \sin\varphi\,\hat{\jmath}}{\sqrt{\cos^2\varphi + \sin^2\varphi}} = \cos\varphi\,\hat{\imath} + \sin\varphi\,\hat{\jmath}$$

$$e_2 = e_\varphi = \frac{\partial r/\partial\varphi}{|\partial r/\partial\varphi|} = \frac{-\rho\sin\varphi\,\hat{\imath} + \rho\cos\varphi\,\hat{\jmath}}{|-\rho\sin\varphi\,\hat{\imath} + \rho\cos\varphi\,\hat{\jmath}|}$$

$$= \frac{\rho(-\sin\varphi\,\hat{\imath} + \cos\varphi)}{\sqrt{\rho^2\cos^2\varphi + \rho^2\sin^2\varphi}} = -\sin\varphi\,\hat{\imath} + \cos\varphi\,\hat{\jmath}$$

$$e_3 = e_z = \frac{\partial r/\partial z}{|\partial r/\partial z|} = \hat{k}$$

Then

$$e_1.e_2 = (\cos\varphi\,\hat{\imath} + \sin\varphi\,\hat{\jmath}).(-\sin\varphi\,\hat{\imath} + \cos\varphi\,\hat{\jmath})$$
$$= -\sin\varphi\cos\varphi + \sin\varphi\cos\varphi = 0$$
$$e_2.e_3 = (-\sin\varphi\,\hat{\imath} + \cos\varphi\,\hat{\jmath}).\hat{k} = 0$$
$$e_3.e_1 = \hat{k}.(\cos\varphi\,\hat{\imath} + \sin\varphi\,\hat{\jmath}) = 0$$

and so e_1, e_2 and e_3 are mutually perpendicular and the coordinate system is orthogonal.

Example 2

Problem Statement

Prove

$$\frac{de_\rho}{dt} = \varphi\dot{e}_\varphi$$
$$\frac{de_\varphi}{dt} = -\dot{\varphi}e_\rho$$

where dots denote differentiation with respect to time t.

Solution

We have

$$e_\rho = \cos\varphi\,\hat{\imath} + \sin\varphi\,\hat{\jmath}$$

and

$$e_\varphi = -\sin\varphi\,\hat{\imath} + \cos\varphi\,\hat{\jmath}$$

Then

$$\frac{de_\rho}{dt} = \frac{d}{dt}(\cos\varphi\,\hat{\imath} + \sin\varphi\,\hat{\jmath})$$
$$= -\sin\varphi\,\dot{\varphi}\hat{\imath} + \cos\varphi\,\dot{\varphi}\hat{\jmath} = \dot{\varphi}(-\sin\varphi\,\hat{\imath} + \cos\varphi\,\hat{\jmath}) = \dot{\varphi}e_\varphi$$
$$\frac{de_\varphi}{dt} = \frac{d}{dt}(-\sin\varphi\,\hat{\imath} + \cos\varphi\,\hat{\jmath})$$
$$= -\cos\varphi\,\dot{\varphi}\hat{\imath} - \sin\varphi\,\dot{\varphi}\hat{\jmath} = -\dot{\varphi}(\cos\varphi\,\dot{\varphi}\hat{\imath} + \sin\varphi\,\dot{\varphi}\hat{\jmath}) = -\dot{\varphi}e_\rho$$

Module No. 114

Introduction to Projectile, Motion of a Projectile

Introduction to Projectile

If a ball is thrown from one person to another or an object is dropped from a moving plane, then their path of traveling/motion is often called a projectile. If air resistance is negligible, a projectile can be considered as a freely falling body so the Eq of motion will be

$$\mathrm{m}\frac{\mathrm{d}^2\mathrm{r}}{\mathrm{dt}^2} = -\mathrm{mg}$$

or

$$\frac{\mathrm{d}^2\mathrm{r}}{\mathrm{dt}^2} = -\mathrm{g}$$

with appropriate I.Cs

Motion of a Projectile with Resistance

Earlier, we studied the motion of a projectile under the assumption that air resistance is negligible. If we further assume that the motion takes place in the vertical plane, (XY-plane), and the only force acting on the projectile is the gravity, then the equation of motion will be

$$\frac{\mathrm{d}^2\mathrm{r}}{\mathrm{dt}^2} = \vec{\mathrm{g}} = -\mathrm{g}\hat{\jmath}$$

Here we will discuss the problem of projectile motion when air resistance is taken into account. Assuming the model of retarding force in which

$F \infty v$

or

$$F = -k_1 v$$

Let the initial velocity of the projectile be v_0 and angle of projection be θ

The initial conditions can be taken as

$$x(t=0) = y(t=0) = 0$$
$$\dot{x}(t=0) = v_0 \cos\theta = u_1$$
$$\dot{y}(t=0) = v_0 \sin\theta = v_1$$

The equation of motion in the horizontal and vertical direction can be written as

$m\ddot{x} = -k_1\dot{x}$

or $\quad \ddot{x} = -\frac{k_1}{m}\dot{x} = -k\dot{x} \qquad (1)$

$m\ddot{y} = -k\dot{y} - mg \qquad (2)$

By the substitution $\dot{x} = z$ we can solve eq (1) as

$$z = \dot{x} = Ae^{-kt}$$

With the initial condition $\dot{x} = u_1$, when $t = 0$, we obtain the constant of integration $A = u_1$ so that

$$\dot{x} = u_1 e^{-kt}$$

Another integration gives

$$x = -\frac{u_1 e^{-kt}}{k} + B$$

The initial condition $x(0) = 0$ gives $B = {}^{u_1}/_{k}$. Hence the solution of equation (1) can be written as

$$x = \frac{u_1}{k}(1 - e^{-kt}) \qquad (3)$$

We can solve equation (2) in the same manner and obtain

$$y = -\frac{gt}{k} + \frac{kv_1 + g}{k^2}(1 - e^{-kt}) \quad (4)$$

The path of motion can be obtained from (3) and (4) by eliminating t

$$y = \frac{g}{k^2}\ln\left(1 - \frac{kx}{u_1}\right) + \frac{kv_1 + g}{k^2}\frac{kx}{u_1}$$

which is no longer a parabola.

We can solve equation (2) in the same manner and obtain

$$y = -\frac{gt}{k} + \frac{kv_1 + g}{k^2}(1 - e^{-kt}) \quad (4)$$

The path of motion can be obtained from (3) and (4) by eliminating t

$$y = \frac{g}{k^2}\ln\left(1 - \frac{kx}{u_1}\right) + \frac{kv_1 + g}{k^2}\frac{kx}{u_1}$$

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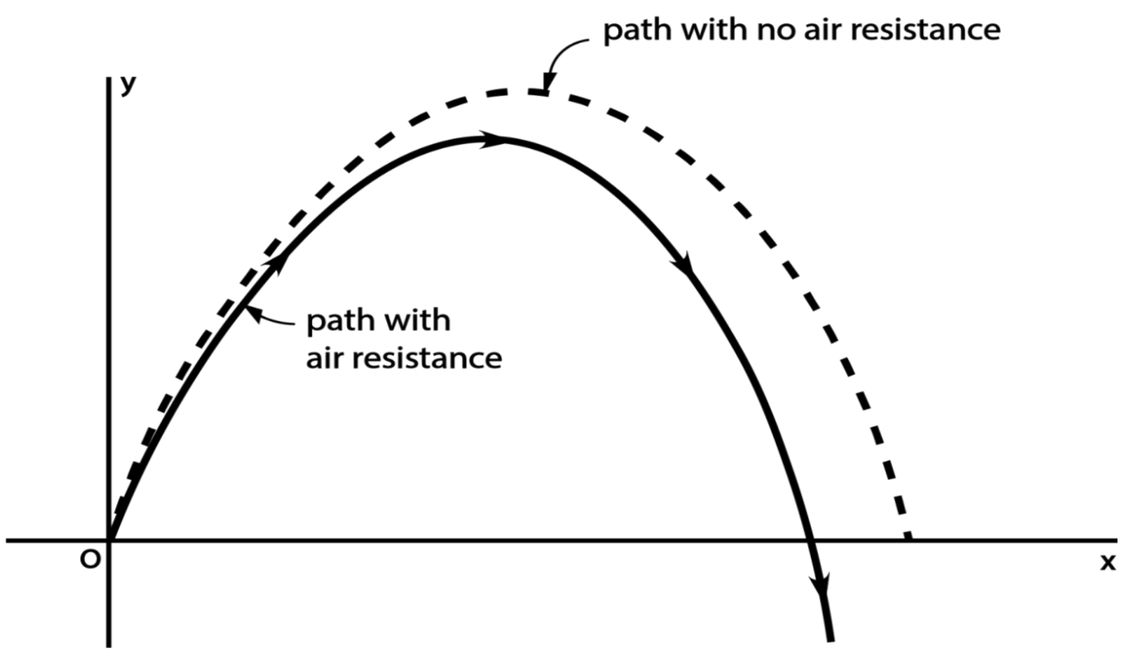


Time of Flight τ

Time of flight can be found by putting in equation (4) $y = 0$ and $t = \tau$

$$0 = -\frac{g\tau}{k} + \frac{kv_1 + g}{k^2}(1 - e^{-k\tau})$$

$$\tau = \frac{kv_1+g}{gk}(1 - e^{-k\tau}) \qquad (5)$$

This equation can be solved exactly for τ. To find an approximate solution, we expand the exponential factor in equation (5), using

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \dots \dots ..$$

Form equation (5)

$$\tau = \frac{kv_1 + g}{gk}\left(k\tau - \frac{k^2\tau^2}{k\tau} + \frac{k^3\tau^3}{3!} + \cdots ..\right)$$

$$1 = \frac{kv_1 + g}{gk}\left(k - \frac{k^2\tau}{k\tau} + \frac{k^3\tau^2}{3!} + \cdots ..\right)$$

which on simplification gives

$$\tau = \frac{2v_1}{kv_1 + g} + \frac{1}{3}k\tau^2 + \cdots ..$$

In the limit of small air resistance (i.e. $k \to 0$), the above expression reduces to

$$\tau = \frac{2v_1}{kv_1+g} + \frac{k}{3}\tau^2 \qquad (6)$$

When there is no resistance, $(k = 0)$, obtain from (6)

$$\tau_0 = \frac{2v_1}{g} = \frac{2v_0 \sin\theta}{g}$$

If k is non-zero small number, then the time of flight τ will be very close to τ_0. Substituting τ_0 for τ on the R.H.S of (6), and simplifying we get

$$\tau = \frac{2v_1}{g}\left(1 - \frac{kv_1}{3g}\right)$$

The above equation gives a formula for approximate time of flight in the presence of a weak retarding force.

Module No. 115

Conservation of Energy for a System of Particles

Statement

"The law of conservation of energy describes that the net energy of an isolated system remains conserved. Energy can neither be created nor destroyed; rather, it transforms from one form to another."

Theorem Statement

(Principle of conservation of energy)

In case of conservative force field, the total energy is a constant. i.e.,

If T is for kinetic energy and V is for potential energy, then the total energy E is

$E = T + V =$ constant

Explanation

For a conservative force field, we have already learned that the work done by the system of particles is

Work done = change in kinetic energy = $W = T_2 - T_1$

Also Work done = change in potential energy = $W = V_1 - V_2$

By comparing, we get

$$T_2 - T_1 = V_1 - V_2$$

or

$$T_1 + V_1 = T_2 + V_2$$

which can be written as

$$\frac{1}{2}m{v_1}^2 + V_1 = \frac{1}{2}m{v_2}^2 + V_2$$

.

Module No. 116

Conservation of Energy

Example

Consider the force field $F = -kr^3\vec{r}$

i. Find whether the given field is conservative or not.

ii. Find the potential energy of the given force field of part (i).

iii. For the particle moving in $xy-$ plane, find the work done by the force in moving the particle from A to B, where A is the point where $r = a$, and B where $r = b$.

iv. If the particle of mass m moves with velocity $v = \frac{dr}{dt}$ in this field, show that if E is the constant, total energy then

v. $\frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{1}{5}kr^5 = E$

Solution

i. As $\vec{F} = -kr^3\vec{r} = -k(x^2+y^2)^{3/2}(xi+yj)$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -k(x^2+y^2)^{3/2}x & -k(x^2+y^2)^{3/2}y & 0 \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}\left(-k(x^2+y^2)^{3/2}y\right)\right] + j\left[\frac{\partial}{\partial z}\left(-k(x^2+y^2)^{3/2}x\right) - \frac{\partial}{\partial x}(0)\right]$$

$$+ k\left[\frac{\partial}{\partial x}\left(-k(x^2+y^2)^{3/2}y\right) - \frac{\partial}{\partial y}\left(-k(x^2+y^2)^{3/2}x\right)\right]$$

$$= -3k(x^2+y^2)^{\frac{1}{2}}xy + 3k(x^2+y^2)^{\frac{1}{2}}xy = 0$$

Thus the given field is conservative.

ii. Since the field is conservative there exists a potential V such that

$$F = -\nabla V$$

$$F = -kr^3\vec{r} = -k(x^2+y^2)^{\frac{3}{2}}(xi+yj)$$

$$= -k(x^2+y^2)^{\frac{3}{2}}xi - k(x^2+y^2)^{\frac{3}{2}}yj = -\nabla V$$

$$= -\frac{\partial V}{\partial x}i - \frac{\partial V}{\partial y}j - \frac{\partial V}{\partial z}k$$

By comparing, we get

$\frac{\partial V}{\partial x} = k(x^2 + y^2)^{3/2}x, \quad \frac{\partial V}{\partial y}=k(x^2 + y^2)^{3/2}y$ and $\frac{\partial V}{\partial z} = 0$

From which, by omitting the constants, we get

$$V = \frac{1}{5}\,k(x^2 + y^2)^{5/2} = \frac{1}{5}\,kr^5$$

is required potential.

iii. The velocity potential at point A

$$= \frac{1}{5}\,ka^5$$

and the velocity potential at point B

$$= \frac{1}{5}kb^5$$

So, the work done from A to B is

= Potential at A - Potential at B

$$= \frac{1}{5}\,ka^5 - \frac{1}{5}\,kb^5 = \frac{1}{2}\,k(a^5 - b^5)$$

which is required work done by the given force field.

iv. Since the kinetic energy of a particle of mass m moving with velocity v $= \frac{dr}{dt}$ is

$$T = \frac{1}{2}\,mv^2 = \frac{1}{2}\,m(\frac{dr}{dt})^2$$

From part (ii), we have the potential energy

$$V = \frac{1}{5}\,kr^5$$

Thus the total energy E will be

$$E = T + V$$

$$= \frac{1}{2}\,m\left(\frac{dr}{dt}\right)^2 + \frac{1}{5}\,kr^5$$

Hence proved.

Module No. 117

Introduction to Impulse – Derivation

Impulse is a special type of force defined by applying the integral of a force F, over the time interval, t, for which it acts on the body.
Impulse is a directional (vector) quantity in the same direction of force as force is also a directional quantity. When Impulse is applied to a rigid body, it results a corresponding vector change in its linear momentum along the same direction. The SI unit of impulse is the newton second ($N \cdot s$), and the dimensionally equivalent unit of momentum is the kilogram meter per second ($kgms^{-1}$).
The particle is located at P_1 and P_2 at times t_1 and t_2 where it has velocities v_1 and v_2 respectively.
The time integral of the force F given by

$$\int_{t_1}^{t_2} F dt$$

is called the impulse of the force F.

Theorem Statement

The impulse is equal to the change in momentum; or, in symbols,

$$\int_{t_1}^{t_2} F dt = mv_2 - mv_1 = p_2 - p_1$$

Proof

We have to prove that the impulse of a force is equal to the change in momentum.
By definition of impulse and Newton's second law, we have

$$\int_{t_1}^{t_2} F dt = \int_{t_1}^{t_2} m \frac{dv}{dt} dt = \int_{t_1}^{t_2} m dv = m|v|_{t_1}^{t_2} = mv_2 - mv_1 = p_2 - p_1$$

where we use the conditions
$v(t_1) = v_1$ and $v(t_2) = v_2$
The theorem is true even when the mass is variable and the force is non-conservative.

Module No. 118

Example of Impulse

Example:

What is the magnitude of the impulse developed by a mass of 200 gm which changes its velocity from $5i - 3j + 7k$ m/sec to $2i + 3j + k$ m/sec?

Solution:

Since we have the following information:

$$m = 200 \text{ gm}$$
$$v_1 = 5i - 3j + 7k$$
$$v_2 = 2i + 3j + k$$

As we know that

$$\text{Impulse} = p_2 - p_1$$

$$= mv_2 - mv_1$$

$$= m(v_2 - v_1)$$

Substituting the values, we get

$$\text{Impulse } = 200(2i + 3j + k - (5i - 3j + 7k))$$

$$= 200(-3i + 6j - 6k)$$

The magnitude of the Impulse will be

$$\text{Impulse magnitude} = 200\sqrt{9 + 36 + 36}$$

$$= 200\sqrt{81}$$

$$= 200(9)$$

$$= 1800 \text{mgm/sec}$$

$$= 1.8 \text{ N sec}$$

Module No. 119

Torque

Definition

Torque is defined as the turning effect of a body. It is trend of an acting force due to which the rotational motion of a body changes. It is also called twist and rotational force on an object. Mathematically, torque is defined as the cross product of the force vector to the distance vector, which causes rotational motion of the body.

$$\boldsymbol{\tau} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}}$$

The magnitude of torque depends upon the applied force, the length of the lever arm connecting the axis to the point where the force applied, and the angle between the force vector and the length of lever arm. Symbolically we can write it as:

$$\boldsymbol{\tau} = |\boldsymbol{r}||\boldsymbol{F}| \sin \boldsymbol{\theta}$$

Torque is a vector quantity implies that it has direction as well as magnitude.
The SI unit for torque is the newton meter (Nm).
The direction of torque can be approximate using **Right Hand Rule.**

Theorem:

The torque acting on a particle equals the time rate of change in its angular momentum, i.e.,

$$\tau = \frac{d\Omega}{dt}$$

where $\Omega = \mathrm{r} \times \mathrm{p}$ is defined angular momentum of the system.

Proof:

As we know torque is defined as

$$\tau = r \times F = r \times ma$$

as m is constant, therefore

$$= m(r \times a)$$

$$= m(r \times \frac{dv}{dt})$$

$$= m\frac{d}{dt}(r \times v)$$

$$= \frac{d}{dt}(r \times mv)$$

$$= \frac{d}{dt}(r \times p)$$

where $p = mv$ is defined as linear momentum and hence the theorem.

Module No. 120

Example of Torque

Example

A particle of mass m moves Along a space curve defined by $r = a\cos\omega t\, i + b\sin\omega t\, j$.

- Find

(i) The Torque

(ii) The angular momentum about the origin.

Solution

(i) As we know that the torque acting on a particle is equal to the time rate of change of its angular momentum, i.e.

$$\tau = \frac{d\Omega}{dt} = \frac{d}{dt}(r \times p)$$

where $p = mv$.

As $r = a\cos\omega t\, i + b\sin\omega t\, j$. Therefore

$$v = \frac{dr}{dt}$$

$$v = -a\omega\sin\omega t\, i + b\omega\cos\omega t\, j$$

So,

$$p = -am\omega\sin\omega t\, i + bm\omega\cos\omega t\, j$$

Now

$$r \times p$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a\cos\omega t & b\sin\omega t & 0 \\ -am\omega\sin\omega t & bm\omega\cos\omega t & 0 \end{vmatrix}$$

$$= [abm\omega\cos^2\omega t + abm\omega\sin^2\omega t]\hat{k}$$

$$= abm\omega\hat{k}$$

$$\tau = \frac{d\Omega}{dt} = \frac{d}{dt}(r \times p) = 0$$

(ii) From part (i), we already have

$$\Omega = r \times p$$

$$= abm\omega\hat{k}$$

is the required angular momentum.

Module No. 121

Introduction to Rigid Bodies and Elastic Bodies

Definition of Rigid Bodies

- When a force is applied to an object/ system of particles, and if the object maintains its overall shape, then the object is called a rigid body.
- Gap between two fixed points on the rigid body remains same regardless of external forces exerted on it.
- We can neglect the deformation of such bodies.
- A rigid body usually has continuous distribution of mass.

Definition of Elastic Bodies

- When a force is applied to a system of particles, it changes the distance between individual particles. Such systems are often called deformable or elastic bodies.

Examples

- A spring and rubber band are some common examples of elastic bodies.
- A wheel is a common example of rigid body.

Module No. 122

Properties of Rigid Bodies

Following are some of the properties of the rigid bodies.

Degree of freedom

The number of coordinates required to specify the position of a system of one or more particles is called the number of degrees of freedom of the system.
For example a particle moving freely in space requires 3 coordinates, e.g. (x, y, z), to specify its position. Thus the number of degrees of freedom is 3.
Similarly, a system consisting of N particles moving freely in space requires 3N coordinates to specify its position. Thus the number of degrees of freedom is 3N.

Translations

A displacement of a rigid body is a direct change of position of its particles. Translational motion is the displacement of all particles of the body by the same amount and the line segment joining the initial and the final position of the particles represented by parallel vectors.
Examples of translational motion are particles freely falling down to earth and the motion of a bullet fired from a gun.

Rotations

Circular motion of a body about a fixed point or axis is called rotation. If during a displacement the points of the rigid body on some line remains fixed and all other are displaced through the same angle, then this displacement is called rotation. A rigid performs rotations around an imaginary line called a rotation axis.
If the axis of rotation passes through the center of mass of the rigid body then body is said to be spin or rotate upon itself. If a body rotates about some external fixed point is called revolution or orbital motion of the rigid body. The example of revolution is the rotation of earth around sun and motion of moon around sun.
Rotational motion concerns only with rigid bodies. The reverse rotation of a body (inverse rotation) is also a rotation.
A wheel is common examples of rotation.

Module No. 123

Instantaneous Axis and Center of Rotation

Introduction to General Plane Motion

The general plane motion of a rigid body can be considered as:

- Translational motion along the given fixed plane and rotational motion about a suitable axis perpendicular to the plane.
- This fixed axis is specifically chosen to pass through the center of mass of the rigid body.

Instantaneous Axis of Rotation

- The axis about which the rigid body rotates is called instantaneous axis of rotation, where this axis is perpendicular to the plane.

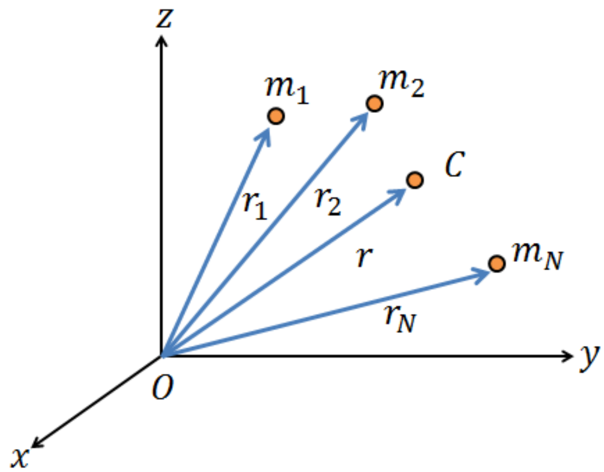
Instantaneous Center of Rotation

- The point where instantaneous axis meets the fixed plane along which the body performs translation motion is described as the instantaneous center of rotation.

Module No. 124

Centre of Mass & Motion of the Center of Mass

The center of mass (c.m.) or centroid of system of particles is a hypothetical particle such that if the entire mass of the system were concentrated there, the mechanical properties would remain the same. In particular expression of linear momentum, angular momentum and kinetic energy assume simpler or more convenient forms when referred to the coordinated of this hypothetical particle and the equation of motion can be reduced to simpler equation of a single particle.
Let $r_1, r_2, r_3, \dots \dots \dots, r_N$ be the position vectors of a system of N particles of masses $m_1, m_2, m_3, \dots \dots \dots, m_N$ respectively [see Fig.].



The center of mass or centroid of the system of particles is defined as that point C having position vector

$$r_c = \frac{\sum_i m_i r_i}{\sum_i m_i}$$

When the system is moving the position vector will depend on time t and therefore r_c will also be a function of t. the velocity and acceleration of center of mass will be then given by

$$\dot{r}_c = \boldsymbol{v_c} = \frac{\sum_i m_i \dot{r}_i}{\sum_i m_i}$$

$$\ddot{r}_c = \boldsymbol{a_c} = \frac{\sum_i m_i \ddot{r}_i}{\sum_i m_i}$$

Motion of Center of Mass

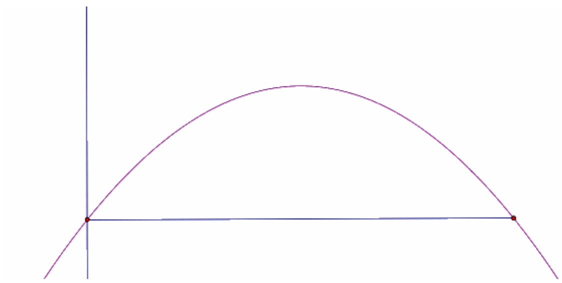
Motion of center of mass can be examined by considering the following points:

1. If a system experiences no external force, the center-of-mass of the system will remain at rest, or will move at constant velocity if it is already moving.

2. If there is an external force, the center of mass accelerates according to $F = ma$.

3. Basically, the center-of-mass of a system can be treated as a point mass, following Newton's Laws.

4. If an object is thrown into the air, different parts of the object can follow quite complicated paths, but the center-of-mass will follow a parabola.



5. If an object explodes, the different pieces of the object will follow seemingly independent paths after the explosion. The center of mass, however, will keep doing what it was doing before the explosion. This is because an explosion involves only internal forces.

Module No. 125

Centre of Mass & Motion of the Center of Mass

The moment of inertia of a rigid body is a property which depends upon its mass and shape, (i.e. the mass distribution of the body) and determines its behavior in rotational motion.
In rotational motion, the moment of inertia plays the same role as the mass in linear motion.

Definition of Moment of Inertia

Formally the moment of inertia I of the particle of mass m about a line is defined by

$$I = md^2$$

where d is the perpendicular distance between the particle and the line (called the axis).

Moment of Inertia of System of particles

The moment of inertia of a system of particles, with masses $m_1, m_2, m_3, \dots, m_N$ about the line or axis AB is defined as

$$I = \sum_{i=1}^{N} m_i \, {d_i}^2$$

$$= m_1{d_1}^2 + m_2{d_2}^2 + \cdots + m_N{d_N}^2$$

In dimensions, the moment of inertia can be expressed as

$$[I] = [M][L^2]$$

Moment of Inertia in Coordinate System

The moment of inertia of a particle of mass m with coordinates (x, y, z) relative to the orthogonal Cartesian coordinate system$OXYZ$ about $X,\ Y,\ Z$ axes will be

$$I_{xx} = m(y^2 + z^2)$$

$$I_{yy} = m(z^2 + x^2)$$

$$I_{zz} = m(x^2 + y^2)$$

Product of Inertia

The product of inertia for the same particle w.r.to the pair of coordinate axes are defined as

$$I_{xy} = -mxy$$

$$I_{yz} = -myz$$

$$I_{zx} = -mzx$$

These definitions can be easily generalized to a system of particle and a rigid body.

Module No. 126

Example of moment of Inertia and Product of Inertia

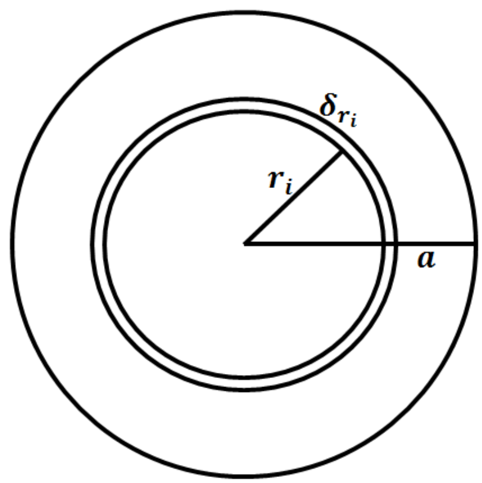
Problem Statement:

Find the moments of inertia of a ring of radius a about an axis through center.

Solution:

Let M be the mass and a the radius of the ring. Then the mass per unit length will be $M/_{2\pi a}$.
We regard this ring to be composed of small elements of mass (δm) each of length δs,
we can write it as

$$\frac{\delta m}{\delta s} = \frac{M}{2\pi a} \Longrightarrow \delta m = \frac{M}{2\pi a}\delta s$$



Moment of inertia of the element about an axis through center O and the perpendicular to the plane of the ring equals $\frac{M}{2\pi a}\delta s\, a^2$.
Therefore the M.I of the whole ring will be

$$I_{ring} = \frac{M}{2\pi a}a^2 \sum\nolimits_{elements} \delta s = \frac{Ma}{2\pi}\int ds = \frac{Ma}{2\pi} \times 2\pi a = Ma^2$$

Hence Ma^2 is the required moment of inertia of the ring.

Module No. 127

Radius of Gyration

Radius of gyration specifies the distribution of the elements of body around the axis in terms of the mass moment of inertia, As it is the perpendicular distance from the axis of rotation to a point mass m that gives an equivalent inertia to the original object m The nature of the object does not affect the concept, which applies equally to a surface bulk mass.
Mathematically the radius of gyration is the root mean square distance of the object's parts from either its center of mass or the given axis, depending on the relevant application.
Let $I = \sum_{i=1}^{N} m_i \, {d_i}^2$ be the moment of inertia of a system of particles about AB, and
$M = \sum_{i=1}^{N} m_i$ be the total mass of the system.
Then the quantity K such that

$$K^2 = \frac{I}{M} = \frac{\sum_{i=1}^{N} m_i \, {d_i}^2}{\sum_{i=1}^{N} m_i}$$

or

$$K = \sqrt{\frac{I}{M}}$$

is called the radius of gyration of the system AB.

Example:

Find the radius of gyration, K, of the triangular lamina of mass M and moment of inertia $I = {}^{1}/_{6}\, Mh^2$.

Solution:

Since formula for radius of gyration is given by

$$K^2 = \frac{I}{M}$$

by substituting value in above , we get

$$K^2 = \frac{{}^{1}/_{6}\, Mh^2}{M} = {}^{1}/_{6}\, h^2$$

or
$$K = \frac{h}{\sqrt{6}}$$

Module No. 128

Principal Axes for the Inertia Matrix

Principal Axes

When a rigid body is rotating about a fixed point O, the angular velocity vector ω and the angular momentum vector L (about O) are not in general in the same direction. However it can be proved that at each point in the body there exists distinct directions, which are fixed relative to the body, along which the two vectors are aligned i.e. coincident. Such directions are called **principal directions** and the axes along them are referred to as **principal axes of inertia.** The corresponding moments of inertia are called principal moments of inertia.

Orthogonality of Principal Axes

If the principal axes at each point of the body exist, then their orthogonality can be proved by stating that axes relative to which product of inertia are zero are the principal axes.

Module No. 129

Introduction to the Dynamics of a System of Particles

Dynamics

Dynamics is the branch of mechanics deals with forces and relationship fundamentally to the motion but sometimes also to the equilibrium of bodies.

The Dynamics of a System of Particles

Suppose we have a system of N particles in motion under forces. These will be two types of forces, internal and external.
Internal forces act between particles of the system; all other are external. We suppose further that the internal forces satisfy Newton's third law of motion. i.e. Internal forces between a pair of particles are equal and opposite If $F_{ij}^{(int)}$ denotes the internal force on the ith particle due to the jth particle, then in the view of this assumption we can write

$$F_{ij}^{(int)} = -F_{ji}^{(int)} \tag{1}$$

The equation of motion for the ith particle of system can therefore be written as

$$F_i^{(ext)} + \textstyle\sum_{j=1}^{N} F_{ij}^{(int)} = m_i \vec{a}_i \tag{2}$$

where $F_i^{(ext)}$ denotes the external force on the ith particle, and $\vec{a}_i$, the acceleration of the ith particle. It is clear from equation (1) that the internal force on a given particle due to itself is zero and therefore the term $i = j$ does not contribute to the sum.
To obtain an equation of motion for the whole system we sum over i, (from 1 to N), on the both sides of equation (2), and obtain

$$\sum_i F_i^{(ext)} + \sum_i \sum_j F_{ij}^{(int)} = \sum_i m_i \vec{a}_i$$

or

$$F^{(ext)} + \textstyle\sum_{i,j} F_{ij}^{(int)} = \sum_i m_i \vec{a}_i \tag{3}$$

Where $F^{(ext)}$ denotes the total external force on the system. Now we will show that because of condition (1), the second term on L.H.S of (3) is zero.

$$\sum_{i,j} F_{ij}^{(int)} = \frac{1}{2} \sum_{i,j} (F_{ij}^{(int)} + F_{ij}^{(int)})$$

Since the indices I and j are dummy and vary over the same set of integers, we can interchange them in the second sum on the R.H.S of the last equation. Therefore we have

$$\sum_{i,j} F_{ij}^{(int)} = \frac{1}{2}\sum_{i,j}(F_{ij}^{(int)} + F_{ji}^{(int)})$$
$$= \frac{1}{2}\sum_{i,j}(F_{ij}^{(int)} - F_{ij}^{(int)}) = 0$$

Therefore the equation (3) becomes

$F^{(ext)} = \sum_i m_i \vec{a}_i$ (4)

In case of centroid, the relation becomes

$$\sum_i m_i \vec{a}_i = M\vec{a}_c$$

where $\ddot{r}_c = \vec{a}_c$ is the acceleration of center of mass (c.m.) and M is the total mass of the system. Hence equation (4) can be written as

$F^{(ext)} = M\vec{a}_c = \frac{d}{dt}(M\vec{v}_c)$ (5)

If $F^{(ext)} = 0$ then $\frac{d}{dt}(M\vec{v}_c) = 0$, which implies that $M\vec{v}_c$ =constant.

We conclude that if external forces are on a system of particles are zero, then its momentum will be a constant of motions or a conserved quantity. Equation (5) describes the translational motion of the system and may be referred to as the translational equation of motion which describes the rotational motion of the system, we take the cross product of both sides of equation (2) with r_i, and obtain

$$r_{i\times}F_i^{(ext)} + \sum_{j=1}^{N} r_i \times F_{ij}^{(int)} = m_i r_i \times \vec{a}_i$$

summing over i

$\sum_i r_{i\times}F_i^{(ext)} + \sum_{i,j=1}^{N} r_i \times F_{ij}^{(int)} = \sum_i m_i r_i \times \vec{a}_i$ (6)

Now we will show that in view of our assumption (1) about internal forces, the second term on L.H.S of equation (6) is zero. On interchanging the dummy indices and using (1) we have

$$\sum_{i,j} r_i \times F_{ij}^{(int)} = \frac{1}{2}\sum_{i,j}(r_i \times F_{ij}^{(int)} + r_j \times F_{ji}^{(int)})$$
$$= \frac{1}{2}\sum_{i,j}(r_i \times F_{ij}^{(int)} - r_j \times F_{ij}^{(int)})$$

$= \frac{1}{2}\sum_{i,j}(r_i - r_j) \times F_{ij}^{(int)}$ (7)

The vector $r_i - r_j$ is in the direction of the line segment joining the particles I and j and therefore it is parallel to F_{ij} . Therefore the last term on R.H.S of equation (7) is zero.

Hence we finally obtain

$$\sum_{i,j} r_i \times F_{ij}^{(ext)} = \sum_i m_i r_i \times \vec{a}_i$$

or

$$\sum_i G_i = G^{ext} = \sum_i m_i r_i \times \vec{a}_i = \frac{d}{dt}(\sum_i m_i r_i \times \vec{v}_i)$$
$$= \frac{d}{dt}(\sum_i r_i \times m_i \vec{v}_i)$$
$$= \frac{d}{dt}\sum_i L_i = \frac{dL}{dt}$$

which is usually written as

$$\frac{dL}{dt} = G^{(ext)}$$

where ${G_i}^{(ext)}$ is the torque or momentum due to external force on the ith particle, and ${G_i}^{(ext)}$ is the torque due to all external forces on the system.

Similarly L is the total angular momentum of the system.

$$\sum_{i,j} r_i \times F_{ij}^{(ext)} = \sum_i G_i = G$$

Thus we have obtained the following two equations of motion for a system of particles.

$\frac{d}{dt}(M\vec{v}_c) = F^{(ext)}$ (8)

and

$\frac{dL}{dt} = G^{(ext)}$ (9)

It follows from equation (8) that if the total external torque on the system is zero, then its angular momentum will be a constant of motion or a conserved quantity.

Module No. 130

Introduction to Center of Mass and Linear Momentum

The rigid bodies are a system of particles in which the position of particles is relatively fixed. We consider such a system of particles in this article and will discuss its center of mass and linear momentum.

Center of Mass

Our general system consists of η particles of masses $m_1, m_2, \dots \dots .., m_n$ whose position vectors are, respectively, $r_1, r_2, \dots \dots .., r_n$. We define the center of mass of the system as the point whose position vector r_{cm} is given by

$$r_{cm} = \frac{m_1 r_1 + m_2 r_2 + m_3 r_3 + \cdots \dots \dots + m_n r_n}{m_1 + m_2 + \cdots \dots + m_n} = \frac{\sum_{i=1}^{n} m_i r_i}{m} \qquad (1)$$

where $m = \sum_{i=1}^{n} m_i$ is the total mass of the system.
The above definition of center of mass leads us to three equations

$$x_{cm} = \frac{\sum_{i=1}^{n} m_i x_i}{m}, y_{cm} = \frac{\sum_{i=1}^{n} m_i y_i}{m}, z_{cm} = \frac{\sum_{i=1}^{n} m_i z_i}{m}$$

which are the rectangular coordinates of the center of mass of the system.

Linear Momentum

We define the linear momentum ρ of the system as the vector sum of the linear momentum of the individual particles, namely,

$$p = \sum_i p_i = \sum_i m_i v_i$$

From equation (1)

$$r_{cm} = \frac{\sum_{i=1}^{n} m_i r_i}{m} \qquad (1)$$

$$\dot{r}_{cm} = v_{cm} = \frac{\sum_{i=1}^{n} m_i v_i}{m}$$

it follows that

$$p = m v_{cm}$$

that is, the linear momentum of a system of particles is the product of the velocity of the center of mass and the total mass of the system.

Module No. 131

Law of Conservation of Momentum for Multiple Particles

Statement

In the absence of the external force, the momentum of the system of particles will be conserved.

Proof

Suppose we have a system of N particles in motion under forces. These will be two types of forces, internal and external.
Suppose now that there are external forces $F_1 F_2, \dots, F_i, , \dots, F_n$ acting on the respective particles. In addition, there may be internal forces of interaction between any two particles of the system. We denote these internal forces by ${F_{ij}}^{(int)}$, meaning the force exerted on particle i by particle j. Internal Forces act between particles of the system; all over the external. We suppose further that the internal force satisfy Newton's third law of motion. i.e. Internal forces between a pair of particles are equal and opposite.
If ${F_{ij}}^{(int)}$ denotes the internal force on the ith particle due to the jth particle, then in the view of this assumption we can write

$$ {F_{ji}}^{(int)} = -{F_{ij}}^{(int)} \tag{1} $$

The equation of motion for the ith particle of the system can therefore be written as

$$ {F_i}^{(ext)} + \sum_{j=1}^{N} {F_{ij}}^{(int)} = m_i a_i = \dot{p}_\iota \tag{2} $$

where ${F_i}^{(ext)}$ denotes the total external force on the ith particle, and a_i the acceleration of the ith particle.
To obtain an equation of motion for the whole system, we sum over I (from 1 to N), on both sides of the equation (2).

$$ \sum_i {F_i}^{(ext)} + \sum_i \sum_{j=1}^{N} {F_{ij}}^{(int)} = \sum_i m_i a_i = \dot{p}_\iota $$

$$ F^{(ext)} + \sum_{i,j} {F_{ij}}^{(int)} = \sum_i \; m_i a_i = \dot{p}_\iota \tag{3} $$

where $F^{(ext)}$denotes the total force of the system.
now we will show that because of equation 1, L.H.S of equation 3 is zero.

$$\dot{p}_\iota = \sum_{i,j} F_{ij}{}^{(int)} = \left(\frac{1}{2}\sum_{i,j} F_{ij}{}^{(int)} + \frac{1}{2}\sum_{i,j} F_{ij}{}^{(int)}\right)$$

since the indices I and j are dummy and vary over the same set of integers, we can interchange them in the second sum on R.H.S of the last equation. Therefore we have

$$\dot{p}_\iota = \sum_{i,j} F_{ij}{}^{(int)} = \left(\frac{1}{2}\sum_{i,j} F_{ij}{}^{(int)} + \frac{1}{2}\sum_{i,j} F_{ji}{}^{(int)}\right)$$

using equation (1), we have

$\sum_{i,j} F_{ij}{}^{(int)} = \left(\frac{1}{2}\sum_{i,j} F_{ij}{}^{(int)} - \frac{1}{2}\sum_{i,j} F_{ji}{}^{(int)}\right)$=0

therefore equation (3) becomes

$$\dot{p}_\iota = F^{(ext)} = \sum_i m_i a_i$$

for centroid, we know that $\sum_i \; m_i a_{\boldsymbol{i}} = m a_c$where $a_c$is the acceleration of centroid.
Hence equation can be written as

$$\dot{p}_\iota = F^{(ext)} = M a_c = \frac{d}{dt}(M v_c)$$

If $F^{(ext)} = 0$ then $\dot{p}_\iota = \frac{d}{dt}(M v_c) = 0 \;\; or \;\; p = M v_c =$ constant

We conclude that if external forces are on a system of particles are zero, then its momentum will be a constant of motion or a conserved quantity.

Module No. 132

Example of Conservation of Momentum

Problem Statement

A particle A, of mass 6 kg, travelling in a straight line at 5 ms^{-1} collides with a particle B, of mass 4 kg, travelling in the same straight line, but in the opposite direction, with a speed of 3 ms^{-1}. Given that after the collision particle A continues to move in the same direction at 1.5 ms^{-1}, what speed does particle B move with after the collision? (© mathcentre 2009)

Given Data

Mass of particle $A = m_1 = 6$kg
Velocity of A before collision $= u_1 =$5ms^{-1}
Velocity of A after collision $= v_1 =$1.5ms^{-1}
Mass of Particle $B = m_2 = 4$kg
Velocity of B before collision $= u_2 = 3ms^{-1}$

Required

Velocity of B after collision $= v_2 = ?$

Solution

It is always useful to depict the collision with the velocities both before and after.
Using the principle of conservation of momentum:

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$
$$6 \times 5 + 4 \times (-3) = 6 \times 1.5 + 4 \times v_2$$
$$9 = 4 \times v_2$$
$$v_2 = {}^{9}/_{4} = 2.3\, m\, s^{-1}$$

So after the collision particle B moves with a speed of 2.3 m s−1, in the same direction as A.

Example 2

Problem Statement

A particle A, of mass 8 kg, collides with a particle B, of mass m2 kg. The velocity of particle A before the collision was $(-1i\ +\ 4j)\ m\ s^{-1}$ and the velocity of particle B before the collision was $(-0.8i\ +\ 1.4j)\ m\ s^{-1}$. Given the velocity of particle A after the collision was $(-2i\ +\ 2j)\ m\ s^{-1}$, and the velocity of particle B was $3j\ m\ s^{-1}$, what was the mass of particle B?

Given Data

Mass of particle A= $m_1 = 8$kg
Velocity of A before collision= $u_1 = (-1i\ +\ 4j)\ m\ s^{-1}$
Velocity of A after collision= $v_1 = (-2i\ +\ 2j)\ m\ s^{-1}$
Velocity of B before collision= $u_2 = (-0.8i\ +\ 1.4j)\ m\ s^{-1}$
Velocity of B after collision= $v_2 =? \, 3j\ m\ s^{-1}$

Required

Mass of Particle B= $m_2 =?$

Solution

It is always useful to depict the collision with the velocities both before and after.
Using the principle of conservation of momentum:

$$m_1u_1\ +\ m_2u_2\ =\ m_1v_1\ +\ m_2v_2$$

$$6\ \times\ 5\ +\ 4\ \times\ (-3)\ =\ 6\ \times\ 1.5\ +\ 4\ \times\ v_2$$

$$9\ =\ 4\ \times\ v_2$$

$$v_2\ =\ {}^{9}/_{4} =\ 2.3\text{ms}^{-1}$$

So after the collision particle B moves with a speed of 2.3 ms^{-1}, in the same direction as A.

Example 2

Problem Statement

A particle A, of mass 8 kg, collides with a particle B, of mass 2 kg. The velocity of particle A before the collision was $(-1i\ +\ 4j)\ \text{ms}^{-1}$ and the velocity of particle B before the collision was $(-0.8i\ +\ 1.4j)\ \text{ms}^{-1}$. Given the velocity of particle A after the collision was $(-2i\ +\ 2j)\ \text{ms}^{-1}$, and the velocity of particle B was $3j\ \text{ms}^{-1}$, what was the mass of particle B?

Given Data

Mass of particle $A = m_1 = 8$kg
Velocity of A before collision = $u_1 = (-1i + 4j)$ms^{-1}
Velocity of A after collision = $v_1 = (-2i + 2j)$ms^{-1}
Velocity of B before collision = $u_2 = (-0.8i + 1.4j)$ms^{-1}
Velocity of B after collision = $v_2 = 3j$ ms^{-1}

Required

Mass of Particle $B = m_2 =$?

Solution

By making use of principle of conservation of momentum:

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

$$8\left(-1i + 4j\right) + m2\left(-0.8i + 1.4j\right)$$

$$= 8\left(-2i + 2j\right) + m_2(3j)$$

$$-8i + 32j + 16i - 16j + m2\left(-0.8i + 1.4j\right) - m_2(3j) = 0$$

$$8i + 16j + m_2\left(-0.8i - 1.6j\right) = 0$$

$$8i + 16j - m_2\left(+0.8i + 1.6j\right) = 0$$

$$8i + 16j = m_2\left(0.8i + 1.6j\right)$$

$$m_2 = \frac{0.8i + 1.6j}{8i + 16j}$$

By rationalizing and solving the above fraction, we obtain

$$m_2 = 10kg$$

Module No. 133

Angular Momentum – Derivation

Angular momentum of a particle of mass m, position vector r and linear momentum p is defined as $r \times p$. It is also called moment of momentum.
Let there be a system consists of N particles with positon vector r_i and the momentum p_i $(i = 1,2,3, \dots \dots, N)$ then the total angular momentum **L** is given by

$$L = \sum_i r_i \times p_i = \sum_i r_i \times (mv_i)$$

To find the relation between L and $\overrightarrow{\omega}$, we proceed as follows:

$$L = \sum_i r_i \times (m_i v_i)$$

$$= \sum_i r_i \times m_i(\vec{\omega} \times r_i)$$

$$= \sum_i m_i[r_i \times (\vec{\omega} \times r_i)]$$

$$= \textstyle\sum_i m_i[(r_i . r_i)\vec{\omega} - (r_i . \vec{\omega})r_i] \qquad (1)$$

Here $(r_i . r_i)\vec{\omega}$ and $(r_i . \vec{\omega})r_i$ are not parallel vectors. If they were parallel then we could easily say that L is parallel to $\vec{\omega}$.

We conclude that in general L and $\vec{\omega}$ are not parallel to each other. To find the relationship between the vectors L and $\vec{\omega}$ we proceed as follows:
Since

$$r_i = x_i\hat{\imath} + y_i\hat{\jmath} + z_i\hat{k}$$

therefore

$$r_i . r_i = {x_i}^2 + {y_i}^2 + {z_i}^2$$

and

$$\vec{\omega} . r_i = \omega_x x_i + \omega_y y_i + \omega_z z_i$$

Therefore substituting these values in equation (1), we have

$$L = \textstyle\sum_i m_i({x_i}^2 + {y_i}^2 + {z_i}^2)\left(\omega_x\hat{\imath} + \omega_y\hat{\jmath} + \omega_z\hat{k}\right) - (\omega_x x_i + \omega_y y_i + \omega_z z_i)(x_i\hat{\imath} + y_i\hat{\jmath} + z_i\hat{k}) \qquad (2)$$

Writing

$$L = L_x\hat{\imath} + L_y\hat{\jmath} + L_z\hat{k} \equiv L_1\hat{\imath} + L_2\hat{\jmath} + L_3\hat{k}$$

and

$$\omega = \omega_x\hat{\imath} + \omega_y\hat{\jmath} + \omega_z\hat{k} \equiv \omega_1\hat{\imath} + \omega_2\hat{\jmath} + \omega_3\hat{k}$$

and comparing the coefficient of $\hat{\imath}$ on both sides of equation (2), we obtain

$$L_x = \sum_i m_i(\,{x_i}^2 + {y_i}^2 + {z_i}^2)\omega_x - (\omega_x x_i + \omega_y y_i + \omega_z z_i)x_i$$

$$= \sum_i m_i[\,{x_i}^2\omega_x + ({y_i}^2 + {z_i}^2)\omega_x - \omega_x {x_i}^2 - \omega_y x_i y_i - \omega_z x_i z_i]$$

$$= \omega_x \textstyle\sum_i m_i\,({y_i}^2 + {z_i}^2) - \omega_x {x_i}^2 - \omega_y \sum_i m_i\, x_i y_i - \omega_z \sum_i m_i\, x_i z_i] \qquad (3)$$

As we studied the definitions of moment of inertia and product of inertia be defined as

$$I_{xx} = \sum_i m_i({y_i}^2 + {z_i}^2)$$

$$I_{yy} = \sum_i m_i({z_i}^2 + {x_i}^2)$$

$$I_{zz} = \sum_i m_i\left({x_i}^2 + {y_i}^2\right)$$

and

$$I_{xy} = -\sum_i m_i x_i y_i$$

$$I_{yz} = -\sum_i m_i y_i z_i$$

$$I_{zx} = -\sum_i m_i x_i z_i$$

Using these definitions and noting that $I_{xy} = I_{yx}$ etc. we can write

$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

Similarly we obtain

$$L_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$$

$$L_z = I_{zx}\omega_x + I_{yz}\omega_y + I_{zz}\omega_z$$

Module No. 134

Angular Momentum in Case of Continuous Distribution of Mass

An actual rigid body consists of very large number of particles and therefore we may suppose that there is a continuous distribution of mass. If $\rho(r)$ denotes density of a volume element dV, surrounding the point r, then the mass of this element will be $\rho(r)dV$.
Hence we can write the expression of angular momentum in case of continuous distribution of mass by

$$I_{11} \equiv I_{xx} = \int \rho(r)(y^2 + z^2)dV$$

where

$$r = (x, y, z) \equiv (x_1, x_2, x_3), \quad dV = dxdydz$$

Similarly the other two moment of inertia are defined as

$$I_{22} \equiv I_{yy} = \int \rho(r)(z^2 + x^2)dV$$

and

$$I_{33} \equiv I_{zz} = \int \rho(r)(x^2 + y^2)dV$$

Similarly the product of inertia are defined as

$$I_{12} \equiv I_{xy} = -\int \rho(r)xydV,$$

$$I_{23} \equiv I_{yz} = -\int \rho(r)yzdV,$$

$$I_{31} \equiv I_{zx} = -\int \rho(r)zxdV$$

By using these definitions of moment of inertia, we can find the expression of angular momentum by using the following result

$$L = L_x\hat{\imath} + L_y\hat{\jmath} + L_z\hat{k}$$

$$L = I_{xx}\omega_x + I_{yy}\omega_y + I_{zz}\omega_z + I_{xy}(\omega_x + \omega_y) + I_{yz}(\omega_y + \omega_z) + I_{zx}(\omega_z + \omega_x)$$

where

$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

$$L_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$$

$$L_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z$$

and $\omega_x, \omega_y, \omega_z$ are the components of angular velocity along (x, y, z).
or

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

where we have used the property of products of inertia that
$I_{xy} = I_{yx}$, $I_{yz} = I_{zy}$ and

$$I_{xz} = I_{zx}$$

The above matrix form can be written as

$$L = I\omega$$

Module No. 135

Law of Conservation of Angular Momentum

Angular Momentum

The angular momentum of a single particle is defined as the cross product of linear momentum and position vector of concerned particle.
Mathematically $L = r \times mv$.

Angular Momentum of System of Particles

The angular momentum L of a system of particles is defined accordingly, as the vector sum of the individual angular momentum, namely,

$$L = \sum_{i=1}^{n} r_i \times m_i v_i \tag{1}$$

Law of Conservation of Angular Momentum

The time rate change of angular momentum in the absence of some external forces is zero. Mathematically, we can write

$$\boldsymbol{\frac{dL}{dt} = 0 \Rightarrow L = \text{constant}}$$

Let us calculate the time derivative of the angular momentum. Using the rule for differentiating the cross product, we find

$$\frac{dL}{dt} = \frac{d}{dt}\left(\sum_{i=1}^{n} r_i \times m_i v_i\right) = \sum_{i=1}^{n}(v_i \times m_i v_i) + \sum_{i=1}^{n}(r_i \times m_i a_i)$$

Now the first term on the right vanishes, because, $v_i \times v_i = 0$ and, because $m_i a_i$ is equal to the total force acting on particle i, we can write

$$\frac{dL}{dt} = \sum_{i=1}^{n}(r_i \times m_i a_i)$$

$$\frac{dL}{dt} = \sum_{i=1}^{n}[r_i \times (\sum_i F_i^{(ext)} + \sum_{i=1}^{n}\sum_{j=1}^{n} F_{ij}{}^{(int)})] \tag{2}$$

$$\frac{dL}{dt} = \sum_{i=1}^{n} r_i \times F_i^{(ext)} + \sum_{i=1}^{n}\sum_{j=1}^{n} F_{ij}{}^{(int)}) \tag{3}$$

where F_i denotes the total external force on particle i, and F_{ij} denotes the (internal) force exerted on particle i by any other particle j. Now the double summation on the right consists of pairs of terms of the form

$(r_i \times F_j) + (r_j \times F_{ji})$ (4)

Denoting the vector displacement of particle j relative to particle i by r_{ij}, we see from the triangle shown in Figure that

$r_{ij} = r_j - r_i$ (5)

Therefore, because F_{ji} = -F_{ij}, expression (3) reduces to

$-r_{ij} \times F_{ij}$ (6)

which clearly vanishes if the internal forces are central, that is, if they act along the lines connecting pairs of particles. Hence, the double sum in Equation (3) vanishes. Now the cross product $(r_i \times F_i)$is the moment of the external force F. The sum $\sum(r_i \times F_i)$ is, therefore, the total moment of all the external forces acting on the system. If we denote the total external torque, or moment of force, by N, Equation (3) takes the form

$$\frac{dL}{dt} = N$$

That is, the time rate of change of the angular momentum of a system is equal to the total moment of all the external forces acting on the system.

If a system is isolated, then $N = 0$, and the angular momentum remains constant in both magnitude and direction:

$$L = \sum_{i=1}^{n} r_i \times m_i v_i = \text{Constant vector} \quad (8)$$

This is a statement of the principle of conservation of angular momentum. It is a generalization for a single particle in a central field.

Module No. 136

Example Related to Angular Momentum

Problem Statement

The moon revolves around the earth so that we always see the same face of the moon.

i. Find how the spin angular momentum and orbital angular momentum of the moon w.r.t. the earth are related?
ii. Find the change in spin angular momentum of moon so that we could see the entire moon's surface during a month.

Solution

i. Let M, R_m denotes the mass and the radius of the moon respectively.

Then its spin angular momentum i.e. angular momentum about its axis of rotation is given by

$$L_s = I\omega_s$$

where I is the moment of inertia and ω_s angular velocity of the moon about its own axis
The moment of inertia of the sphere is

$$I = \frac{2}{5}mr^2$$

by generalizing it for moon, we obtain the moment of inertia for the moon

$$I = \frac{2}{5}MR_m{}^2$$

then the angular momentum of moon will be

$$L_s = \frac{2}{5}MR_m{}^2\omega_s \qquad (1)$$

In addition to spinning about its own axis, the moon is also performing orbital motion about the earth. If we denote the orbital angular momentum by L_0, then

$$L_0 = R \times (MV) = R \times (MR\omega_0)$$

$$= MR^2\omega_0 \qquad (2)$$

$$\therefore v = \omega r$$

where we have treated the moon as a particle and R is the distance between the moon and the earth. Since we always see the same face of the moon, the moon makes on rotation about its axis in the same time as it makes on revolution around the earth i.e. $\omega_s = \omega_0$. Form equation (1) and (2) we have

$$\frac{L_s}{L_0} = \frac{\frac{2}{5}MR_m{}^2\omega_s}{MR^2\omega_0}$$

$$= \frac{2}{5}\left(\frac{R_m}{R}\right)^2$$

ii. If we have to see the entire moon's surface during a month, then L_s must either increase or decrease by one-half of its present value.

Module No. 137

Kinetic Energy of a System about Principal Axes – Derivation

Kinetic Energy

The kinetic energy for a particle is given by the following scalar equation:

$$T = \frac{1}{2}mv^2$$

Where:

- T is the kinetic energy of the particle with respect to ground (an inertial reference frame)
- m is the mass of the particle
- v is the velocity of the particle, with respect to ground

Kinetic Energy of a Rigid Body

For a rigid body experiencing planar (two-dimensional) motion, the kinetic energy is given by the following general scalar equation:

$$T = \frac{1}{2}m{v_c}^2 + \frac{1}{2}I_c\omega^2 \qquad (1)$$

where the first term in equation (1) shows the kinetic energy due to the motion of center of mass and second term shows the rotational kinetic energy and subscript c denotes the center of mass and v_c denotes the velocity of c.m and I_c denotes the inertia matrix w.r.t c.m.

Kinetic Energy of a Rigid Body w.r.t Origin

If the rigid body is rotating about a fixed point *O* that is attached to ground, we can express the kinetic energy as:

$$T = \frac{1}{2}I_0\omega^2$$

Where:

I_0 is the moment of inertia of the rigid body about an axis passing through the fixed point *O*, and perpendicular to the plane of motion and ω is the angular velocity.

Kinetic Energy of a Rigid Body about Principal Axes

If the center of mass is oriented along with origin then the coordinate axes (xyz axes) and the principal axes of the rigid body are aligned. In this case, the inertia matrix reduces, i.e. products of inertia are zero along the principal axes, $(I_{xy} = I_{yz} = I_{zx} = 0)$.
For general three-dimensional motion, the kinetic energy of a rigid body about principal axes is given by the following general scalar equation:

$$T = \frac{1}{2} m{v_0}^2 + \frac{1}{2} I_x \omega_x^2 + \frac{1}{2} I_y \omega_y^2 + \frac{1}{2} I_z \omega_z^2 \quad (3)$$

where I_x, I_y, I_z are the moment of inertia along principal axes (xyz axes) also called principal moment of inertia and products of inertia I_{xy}, I_{yz}, I_{zx} are zero along the axes

Equation (3) is the required expression of Kinetic Energy along principal axes.

If the rigid body has a fixed point *O* that is attached to ground, we can give an alternate scalar equation for the kinetic energy of the rigid body:

$$T = \frac{1}{2} I_x \omega_x^2 + \frac{1}{2} I_y \omega_y^2 + \frac{1}{2} I_z \omega_z^2$$

in this case, the kinetic energy due to the motion of c.m. vanishes.

Module No. 138

Moment of Inertia of a Rigid Body about a Given Line

Let M be the mass of the system and $\hat{e}$ a unit vector along the line l. Then $\hat{e} = \lambda\hat{\imath} + \mu\hat{\jmath} + \upsilon\hat{k}$, where $(\lambda, \mu, \upsilon)$are direction cosines of the line.
If I_l denotes the record moment of inertia, then

$$I_l = \sum_i m_i d_i^2$$

Form the figure

$$d_i = |OP| \sin\theta_i = |r_i| \sin\theta_i = r_i \sin\theta_i = |e_i \times r_i|$$

Therefore

$$I_l = \sum_i m_i |e_i \times r_i|^2 \tag{1}$$

Now

$$|e_i \times r_i| = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \lambda & \mu & \upsilon \\ x_i & y_i & z_i \end{vmatrix}$$

$$= (\mu z_i - \upsilon y_i)\hat{\imath} + (\upsilon x_i - \lambda z_i)\hat{\jmath} + (\lambda z_i - \upsilon x_i)\hat{k}$$

$$|e_i \times r_i|^2 = (\mu z_i - \upsilon y_i)^2 + (\upsilon x_i - \lambda z_i)^2 + (\lambda y_i - \mu x_i)^2$$

Hence on substitution in equation (1), we have

$$I_l = \sum_i m_i[\,(\mu z_i - \upsilon y_i)^2 + (\upsilon x_i - \lambda z_i)^2 + (\lambda y_i - \mu x_i)^2]$$

$$I_l = \sum_i m_i[\,\mu^2 z_i{}^2 + \upsilon^2 y_i{}^2 - 2\mu\upsilon y_i z_i + \upsilon^2 x_i{}^2 + \lambda^2 z_i{}^2 - 2\upsilon\lambda x_i z_i + \lambda y_i + \mu x_i - 2\mu\lambda x_i y_i]$$

$$= \sum_i m_i[\,\mu^2(x_i{}^2 + z_i{}^2) + \upsilon^2(x_i{}^2 + y_i{}^2) + \lambda^2(y_i{}^2 + z_i{}^2) - 2\mu\lambda x_i y_i - 2\mu\upsilon y_i z_i - 2\upsilon\lambda x_i z_i$$

$$= \sum_i m_i[\,\lambda^2(y_i{}^2 + z_i{}^2) + \mu^2(x_i{}^2 + z_i{}^2) + \upsilon^2(x_i{}^2 + y_i{}^2) - 2\mu\lambda x_i y_i - 2\mu\upsilon y_i z_i - 2\upsilon\lambda x_i z_i$$

$$= \lambda^2 \sum_i m_i[\,(y_i{}^2 + z_i{}^2) + \mu^2 \sum_i m_i\,(x_i{}^2 + z_i{}^2) + \upsilon^2 \sum_i m_i\,(x_i{}^2 + y_i{}^2) + 2\mu\lambda(-\sum_i m_i\,x_i y_i)$$

$$+ 2\mu\upsilon(-\sum_i m_i\,y_i z_i) + 2\upsilon\lambda(-\sum_i m_i\,x_i z_i)$$

or finally,

$$I_l = \lambda^2 I_{xx} + \mu^2 I_{yy} + \upsilon^2 I_{zz} + 2\mu\lambda I_{xy} + 2\mu\upsilon I_{yz} + 2\upsilon\lambda I_{zx}$$

which may also be written as

$$I_l = \lambda^2 I_{11} + \mu^2 I_{22} + \upsilon^2 I_{33} + 2\mu\lambda I_{12} + 2\mu\upsilon I_{23} + 2\upsilon\lambda I_{31}$$

Module No. 139

Example of M.I of a Rigid Body About Given Line

Problem Statement

Calculate the moment of inertia of a right circular cone about its axis of symmetry.

Solution

Let M be the mass, a the radius and h the height of right circular cone. We regard the cone as composed of elementary circular discs of small thickness each parallel to the base of the cone. We choose the z-axis along the axis of symmetry, and consider a typical disc of radius r and width δz at a distance z from the base.
Mass of the disc is given by

$$\delta m = \rho \pi r^2 \delta z$$

We regard the disk to be composed of concentric elementary circular rings of varying radii say r' then the mass $\delta m'$ of one circular ring with height δz will be

$$\delta m' = \rho 2\pi r' \delta z$$

Then the M.I of one circular ring will be

$$I_{c.r} = \rho 2\pi r' \delta z (r')^2 = 2\pi \rho (r')^3 \delta z$$

Hence the M.I of the whole disk will be

$$\delta I = \sum_{rings} 2\pi \rho (r')^3 \delta z$$

$$= \int_0^r 2\pi \left(\frac{\delta m}{\pi r^2 \delta z}\right)(r')^3 \delta z dr'$$

$$= \frac{2\delta m}{r^2} \int_0^r (r')^3 dr'$$

$$\delta I = \frac{2\delta m}{4r^2} r^4 = \frac{1}{2} \delta m r^2$$

From the similar triangles,

we have

$$\frac{r}{a} = \frac{h-z}{h} \quad \text{or} \quad r = a\frac{h-z}{h}$$

Therefore

$$\delta m = \rho\pi\left(a\frac{h-z}{h}\right)^2 \delta z$$

Since the M.I of the disk is

$$\delta I = \frac{1}{2}\delta m r^2$$

On substituting for δm and r, the M.I of the cone about its axis of symmetry will be

$$I = \frac{1}{2}\int_0^h \rho\pi\left(a\frac{h-z}{h}\right)^2 \left(a\frac{h-z}{h}\right)^2 dz$$

$$= \frac{1}{2}\frac{\rho\pi a^4 h}{h^4}\int_0^h (h-z)^4 dz = \frac{1}{2}\frac{\rho\pi a^4 h}{h^4}\int_0^h (z-h)^4 dz$$

Since the M.I of the disk is

$$\delta I = \frac{1}{2}\delta m r^2$$

On substituting for δm and r, the M.I of the cone about its axis of symmetry will be

$$I = \frac{1}{2}\int_0^h \rho\pi\left(a\frac{h-z}{h}\right)^2 \left(a\frac{h-z}{h}\right)^2 dz$$

$$= \frac{1}{2}\frac{\rho\pi a^4 h}{h^4}\int_0^h (h-z)^4 dz = \frac{1}{2}\frac{\rho\pi a^4 h}{h^4}\int_0^h (z-h)^4 dz$$

$$I = \frac{\rho\pi a^4 h}{10h^4}\left|(z-h)^5\right|_0^h$$

$$= \frac{\rho\pi a^4 h^6}{10h^4} = \frac{\rho\pi a^4 h^2}{10}$$

Since we know that

$$\rho = \text{density of the cone}$$

$$= \frac{M}{(1/3)\pi a^2 h}$$

So,

$$I = \frac{3}{10} M a^2$$

Module No. 140

Ellipsoid of Inertia

We have obtained the expression of moment of inertia of the given line l in terms of moment and product of inertia w.r.t the coordinate axes OXYZ coordinate system whose origin O lies on the line l.

$$I_l = I = \lambda^2 I_{11} + \mu^2 I_{22} + \upsilon^2 I_{33} + 2\mu\lambda I_{12} + 2\mu\upsilon I_{23} + 2\upsilon\lambda I_{31} \quad (1)$$

For the ellipsoid of inertia, we chose a point P such that $|OP| = {}^{1}/_{\sqrt{I}}$. If (x, y, z) are coordinates of point P then

$$\lambda = \frac{x}{|OP|} = x\sqrt{I}, \qquad \mu = \frac{y}{|OP|} = y\sqrt{I}\,, \qquad \nu = \frac{z}{|OP|} = z\sqrt{I}$$

On eliminating λ, μ, ν from equation (1), we obtain

$$I = (x\sqrt{I})^2 I_{11} + (y\sqrt{I})^2 I_{22} + (z\sqrt{I})^2 I_{33} + 2(xy\sqrt{I}\sqrt{I})I_{12} + 2(yz\sqrt{I}\sqrt{I})I_{23} + 2(xz\sqrt{I}\sqrt{I})I_{31}$$

$$I = I[x^2 I_{11} + y^2 I_{22} + z^2 I_{33} + 2xyI_{12} + 2yzI_{23} + 2zxI_{31}]$$

$$1 = x^2 I_{11} + y^2 I_{22} + z^2 I_{33} + 2xyI_{12} + 2yzI_{23} + 2zxI_{31}$$

which can also be written as

$$I_{11}x^2 + I_{22}y^2 + I_{33}z^2 + 2I_{12}xy + 2I_{23}yz + 2I_{31}zx = 1$$

Since I_{11}, I_{22}, I_{33} are all positive, equation (2) represents an ellipsoid. This ellipsoid is called ellipsoid of inertia or momental ellipsoid. The momental ellipsoid contains information about moments or product of inertia at given point. If P is any point on the ellipsoid of inertia, then $|OP| = {}^{1}/_{\sqrt{I}}$ or $I = {}^{1}/_{OP^2}$ i.e. the moment of inertia of a rigid body about any line $\overrightarrow{OP}$ is equals to the reciprocal of the square of the length of $|\overrightarrow{OP}|$.

Module No. 141

Rotational Kinetic Energy

Introduction to Kinetic Energy

Kinetic energy is the energy produced in any body during its motion. It is equal to the half of the product of mass and square of the velocity of the moving body;

$$K.E. = T = \frac{1}{2}mv^2$$

We consider a rigid body in a general state of motion in which it has both translation and rotation w.r.t a fixed coordinate system. We suppose that it has an instantaneous angular velocity ω about a reference point C. We will use a model of a rigid body in which it is considered a collection of large number N of particles which satisfy the constraint of rigidity.
If v is the velocity of C, then the velocity v_i of the ith particle is given by

$$v_i = v + \omega \times r_i$$

If m_i is the mass and v_i the velocity of the ith particle, then the kinetic energy of the ith particle will be

$$T_i = \frac{1}{2}m_i v_i^2$$

therefore the total kinetic energy of the system will be given by

$$T = \sum_{i=1}^{N} T_i = \sum_{i=1}^{N} \frac{1}{2}m_i v_i^2 = \sum_{i=1}^{N} \frac{1}{2}m_i(v + \omega \times r_i)^2$$

$$= \frac{1}{2}\sum_{i=1}^{N} m_i[v^2 + 2v.\omega \times r_i + (\omega \times r_i)^2]$$

$$= \frac{1}{2}\left(\sum_{i=1}^{N} m_i\right)v^2 + \frac{1}{2}\sum_{i=1}^{N} m_i\, 2v.\omega \times r_i + \frac{1}{2}\sum_{i=1}^{N} m_i(\omega \times r_i)^2$$

$$= \frac{1}{2}Mv^2 + v.(\omega \times \sum_{i=1}^{N} m_i r_i) + \frac{1}{2}\sum_{i=1}^{N} m_i(\omega \times r_i)^2$$

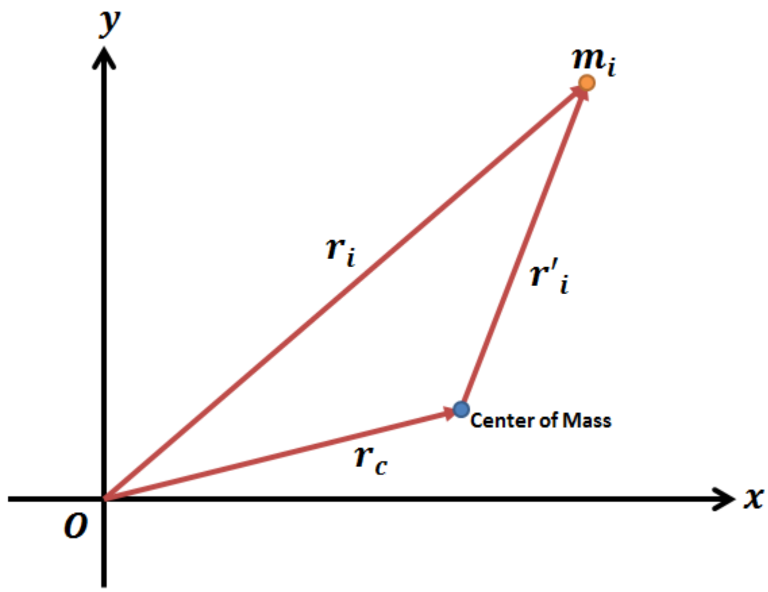
$= \frac{1}{2}Mv^2 + v.\omega \times \sum_{i=1}^{N} m_i r_i + \frac{1}{2}\sum_{i=1}^{N} m_i(\omega \times r_i)^2$ (1)

where $\sum_{i=1}^{N} m_i = M$ is the total mass of the system.
Now by using the definition of position vector center of mass (c.m.), we have

$r_c = \frac{\sum_i m_i r_i}{\sum_i m_i}$ (2)

Now we will discuss the consequence of referring of the position vectors of particle of the system to the c.m. If the position vector of the ith particle of the system w.r.t the c.s. , then

$$r_i = r'_i + r_c$$



Then by substitution this expression in the definition of c.m.in (2) , we obtain

$$r_c = \frac{\sum_i m_i(r'_i + r_c)}{\sum_i m_i}$$

$$r_c = \frac{\sum_i m_i r'_i}{\sum_i m_i} + \frac{Mr_c}{M}$$

$$r_c = \frac{\sum_i m_i r'_i}{\sum_i m_i} + r_c$$

or we can write

$$r_c = \frac{\sum_i m_i r'_i}{M} + r_c$$

which gives

$$\sum_i m_i r'_i = 0$$

Hence if the reference point C is identified with the center of the mass and origin is taken thereat, the expression $\sum_{i=1}^{N} m_i r_i = 0$ and therefore

$$T = \frac{1}{2}Mv^2 + \frac{1}{2}\sum_{i=1}^{N} m_i(\omega \times r_i)^2$$

$$T = T_{tr} + T_{rot}$$

where $T_{tr} = \frac{1}{2}Mv^2$ is the translational kinetic energy is also equals to the K.E of the center of the mass and $t_{rot} = \frac{1}{2}\sum_{i=1}^{N} m_i(\omega \times r_i)^2$ is the rotational kinetic energy of the system.

But we have

$$(\omega \times r_i)^2 = (\omega \times r_i).(\omega \times r_i)$$
$$= \omega.r_i \times (\omega \times r_i)$$

therefore on substitution

$$T_{rot} = \frac{1}{2}\sum_{i=1}^{N} m_i[\omega. r_i \times (\omega \times r_i)] = \frac{1}{2}\omega. L$$

Thus we have obtained the following formulas for translational and rotational K.E. of a rigid body

$$T_{tr} = \frac{1}{2}Mv^2$$

$T_{rot} = \frac{1}{2}\omega. L$ (1)

As we studied the relation $L = I\omega$

By using this value in relation (1) we obtain the rotational kinetic energy in terms of moment of inertia.

$T_{rot} = \frac{1}{2}\omega. I\omega = \frac{1}{2}I\omega^2$ (2)

where I is the moment of inertia of the rigid body about the origin and equation (2) is the required expression for **Rotational Kinetic Energy**.

Module No. 142

Moment of Inertia & Angular Momentum in Tensor Notation

To show that the 3×3 inertia matrix (I_{ij}) is also a Cartesian tensor of rank 3, we proceed as follows. Here we have to distinguish between the particle index and the component index, which denotes particle number, will be denoted by Greek letter α, and will take the value $1,2, \dots \dots, N$. On the other hand, the component index, which denotes the component number will be denotes by the Latin letters such as i and will take the value 1,2,3.
Using this notation, we can write the angular momentum of the system of N particles as

$$L = \sum_\alpha r_\alpha \times p_\alpha = \sum_\alpha r_\alpha \times (m_\alpha v_\alpha)$$

where v_α is the velocity of αth particle. Continuing we have

$$\sum_\alpha m_\alpha r_\alpha \times v_\alpha = \sum_\alpha m_\alpha r_\alpha \times (\omega \times r_\alpha), \qquad \therefore v_\alpha = \omega \times r_\alpha$$

Recall that every particle of the rigid body has the same angular velocity at a given t.
Simplification of the above reduces to

$$L = \sum_\alpha m_\alpha((r_\alpha . r_\alpha)\omega - (\omega . r_\alpha) r_\alpha)$$

$$= \sum_\alpha m_\alpha ({r_\alpha}^2 \omega - (\omega . r_\alpha) r_\alpha)$$

But

$$\omega . r_\alpha = \sum_{i=1}^{3} \omega_i r_{\alpha,i} = \sum_{i=1}^{3} \omega_i x_{\alpha,i} = \sum_{j=1}^{3} \omega_j x_{\alpha,j}$$

where

$$r_\alpha = (x_\alpha, y_\alpha, z_\alpha) = (x_{\alpha,1}, x_{\alpha,2}, x_{\alpha,3}) \equiv x_{\alpha,i}$$

Making substitution and taking the i^{th} component of the expression of angular momentum, we have

$$L_i = \sum_\alpha m_\alpha \left({r_\alpha}^2 \omega_i - (\sum_j \omega_j \, x_{\alpha,j}) x_{\alpha,i} \right)$$

But $\omega_i = \sum_j \; \omega_j \, \delta_{ij}$; where $\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$, thus therefore

$$L_i = \sum_{\alpha} m_{\alpha} \left({r_{\alpha}}^2 \sum_{j} \omega_j \, \delta_{ij} - \sum_{j} \omega_j \, x_{\alpha,j} x_{\alpha,i} \right)$$

$$L_i = \sum_{\alpha} m_{\alpha} \sum_{j} \left({r_{\alpha}}^2 \delta_{ij} - x_{\alpha,j} x_{\alpha,i} \right) \omega_j$$

$$L_i = \sum_{j} \omega_j \sum_{\alpha} m_{\alpha} \left({r_{\alpha}}^2 \delta_{ij} - x_{\alpha,j} x_{\alpha,i} \right)$$

which can also be expressed as

$L_i = \sum_j \; \omega_j \, I_{ij}$ (1)

where

$I_{ij} = \sum_{\alpha} m_{\alpha} \left({r_{\alpha}}^2 \delta_{ij} - x_{\alpha,j} x_{\alpha,i} \right)$ (2)

Equation (1) shows that the component of angular momentum depends not only on the angular velocity ω but also on the inertia tensor I_{ij}.

Since the angular velocity (ω_i) and the angular momentum (L_i) are known to be cartesion tensor of rank 1, it follows from the quotient theorem I_{ij} that I_{ij} is a tensor of rank 2. It is called the inertia tensor.

Module No. 143

Introduction to Special Moments of Inertia

In this module, we will study some special moment of inertia. In order to introduce the moment of inertia of some specific shapes and special rigid bodies, we assume that the rigid body is of uniform density of particles.

Solid Circular Cylinder

We assume the radius of cylinder is $\boldsymbol{a}$ and mass M about axis of cylinder.

$$I = \frac{1}{2}Ma^2$$

Hollow Circular Cylinder

We assume the radius of cylinder is $\boldsymbol{a}$ and mass M about axis of cylinder.
We consider the thickness of Wall of cylinder is negligible.

$$I = Ma^2$$

Solid Sphere

We assume the radius of sphere is $\boldsymbol{a}$ and mass M about a diameter.

$$I = \frac{2}{5}Ma^2$$

Hollow Sphere

We assume the radius of sphere is $\boldsymbol{a}$ and mass M about a diameter.
We consider the thickness of sphere is negligible.

$$I = Ma^2$$

Rectangular Plate

We consider sides of length $\boldsymbol{a}$ and $\boldsymbol{b}$, and mass M about an axis perpendicular to the plate through the center of mass.

$$I = \frac{1}{12}M(a^2 + b^2)$$

Thin Rod

We assume the length of rod is $\boldsymbol{a}$ and mass M about an axis perpendicular to the rod through the center of mass.

$$I = \frac{1}{12} Ma^2$$

Triangular Lamina

We assume the height $\boldsymbol{h}$ and mass **M** of lamina.

$$I = \frac{1}{6} Mh^2$$

Right Circular Cone

We assume the radius of the circular cone is a and mass M.

$$I = \frac{3}{10} Ma^2$$

Module No. 144

M.I. of the Thin Rod – Derivation

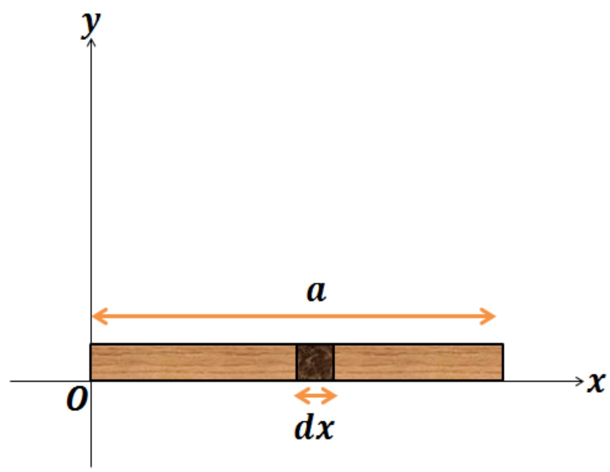
Problem Statement

Calculate the moment of inertia of a uniform rod of length l about an axis perpendicular to the rod and passing through an end point.

Solution

Let the $X-axis$ be chosen along the length of the rod, with origin at one end point as shown in the figure. Let M and a be the mass and length of rod respectively. We suppose the rod to be composed of small elements.

Let dm and dx be the mass and the length of the specific element of the rod at a distance x from the end point O.



Then

$$\frac{dm}{dx} = \frac{M}{a}$$

$$\Longrightarrow dm = \frac{M}{a}dx$$

Then the moment of inertia of this element about the given axis is

$$I_{element} = dmx^2 = \frac{M}{a}x^2dx$$

Hence the moment of inertia of the whole rod will be

$$I = \sum_{all\ elements} \frac{M}{a} x^2 dx$$

$$= \frac{M}{a} \int_0^a x^2 dx$$

$$= \frac{M}{a} . \left| \frac{x^3}{3} \right|_0^a = \frac{M}{a} \frac{a^3}{3}$$

$$= \frac{1}{3} M a^2$$

Module No. 145

M.I. of Hoop or Circular Ring – Derivation

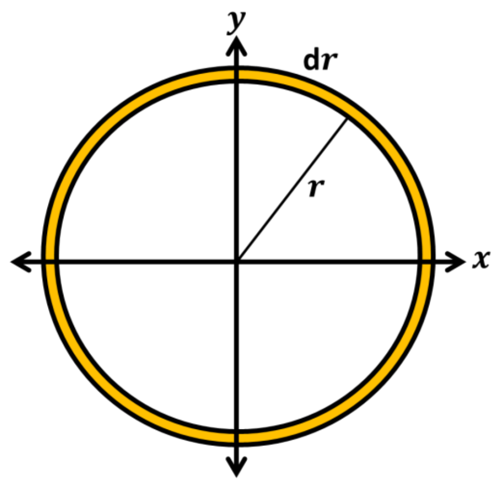
Problem Statement

Calculate the moment of inertia of a hoop of mass M and radius r about an axis passing through its center.

Proof

Let M be the mass and r the radius of the hoop. Then we can define the density of the hoop by

$$\rho = \frac{\text{mass}}{\text{area}} = \frac{M}{2\pi r dr}$$



We consider this hoop to be composed of small masses (δm) each of length δs.
We can write it as

$$\rho = \frac{M}{2\pi r dr} = \frac{\delta m}{dr\delta s}$$

$$\Longrightarrow \delta m = \frac{M}{2\pi r}\delta s$$

Moment of inertia of the small portion of the hoop of mass δm about an axis through center and perpendicular to the plane of the ring equals

$$I_{particle} = \delta m r^2$$

$$= \frac{M}{2\pi r}\delta s\, r^2 = \frac{Mr}{2\pi}\delta s$$

Therefore the $M.I$ of the whole ring/hoop will be

$$I = \frac{Mr}{2\pi} \sum_{particles} \delta s$$

$$I = \frac{Mr}{2\pi} \int ds$$

$$= \frac{Mr}{2\pi} 2\pi r$$

$$= Mr^2$$

- Hence we obtain Mr^2 as the moment of inertia of the hoop.

Module No. 146

M.I. of Annular Disk - Derivation

Problem

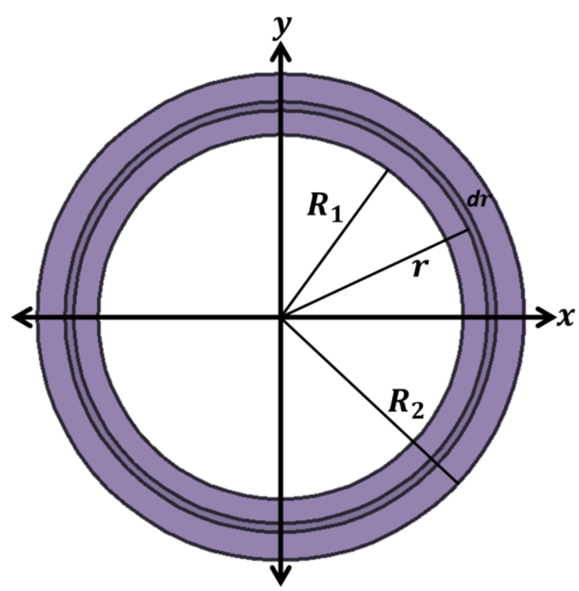
Calculate the moment of inertia of annular disk of mass M. The inner radius of the annulus is R_1 and the outer radius is R_2 about an axis passing through its center.

Solution

Subdivide the annulur disk into concentric rings one of which is shown in the fig.

Let the mass of the ring is dm, and the radius be r, then the moment of inertia of the ring will be:

$$I_{ring} = r^2 dm$$



The Surface area of the ring is

$$\text{Area} = (2\pi r)dr = 2\pi r dr$$

Since the surface area of the

annulus is

$$\pi(R_2^2 - R_1^2)$$

Therefore, we can have

$$\frac{dm}{M} = \frac{2\pi r dr}{\pi(R_2^2 - R_1^2)}$$

$$dm = \frac{2rdr}{(R_2^2 - R_1^2)} M$$

Since the moment of Inertia of the ring is:

$$I_{ring} = r^2 dm$$

or

$$I_{ring} = r^2 \frac{2rdr}{R_2^2 - R_1^2} M = \frac{2Mr^3 dr}{R_2^2 - R_1^2}$$

Thus the total M.I of the annulur disk will be

$$I = \int_{r=R_1}^{R_2} I_{ring}$$

$$I = \int_{r=R_1}^{R_2} \frac{2Mr^3 dr}{R_2^2 - R_1^2}$$

$$= \frac{2M}{R_2^2 - R_1^2} \int_{r=R_1}^{R_2} r^3 dr$$

$$= \frac{2M}{R_2^2 - R_1^2} \left| \frac{r^4}{4} \right|_{R_1}^{R_2}$$

$$= \frac{2M}{R_2^2 - R_1^2} \frac{R_2^4 - R_1^4}{4}$$

$$I = \frac{1}{2} M(R_2^2 + R_1^2)$$

Module No. 147

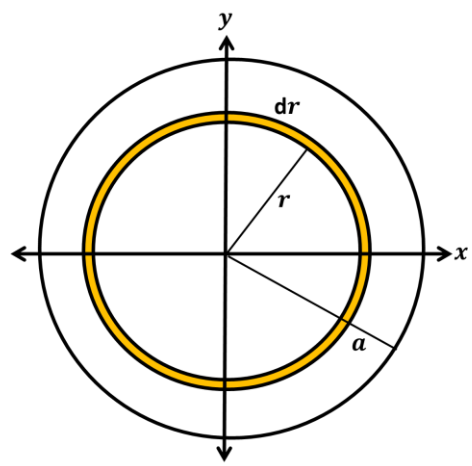
M.I. of a Circular Disk - Derivation

Problem

To find the moment of inertia of a circular disk of radius a, and mass M about the axis of the disk.

Solution

Subdivide the disk into concentric rings one of which is the element (ring) shown in the fig.



Let the mass the of the ring is dm, then the moment of inertia of the ring will be:

$$I_{ring} = r^2 dm$$

The Surface area this element is

$$\text{Area} = (2\pi r)dr = 2\pi r dr$$

Since we have

$$\frac{dm}{M} = \frac{2\pi r dr}{\pi a^2}$$

$$dm = \frac{2rdr}{a^2}M$$

Since the moment of Inertia of the ring is:

$$I_{ring} = r^2 dm$$

or

$$I_{ring} = r^2 \frac{2rdr}{a^2} M = \frac{2Mr^3 dr}{a^2}$$

Thus the total moment of inertia of the circular disk will be

$$I_{disk} = \int_{r=0}^{a} I_{ring}$$

$$I_{disk} = \int_{r=0}^{a} \frac{2Mr^3 dr}{a^2}$$

$$= \frac{2M}{a^2} \int_{r=0}^{a} r^3 dr$$

$$= \frac{2M}{a^2} \left| \frac{r^4}{4} \right|_0^a = \frac{2M}{a^2} \frac{a^4}{4}$$

$$I_{disk} = \frac{1}{2} M a^2$$

which is M.I of a disk.

Module No. 148

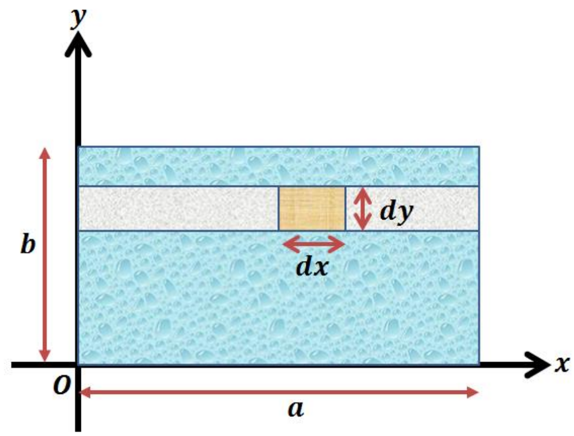
Rectangular Plate – Derivation

Problem Statement

Calculate the inertia matrix of a uniform rectangular plate with sides $\boldsymbol{a}$ and $\boldsymbol{b}$ about its side.

Solution

We consider a rectangular plate (lamina) of sides of length a and b. We consider an element of length dx and dy.



The mass of selected element will be

$$dm = \rho dxdy$$

Its moment of inertia about the y-axis is

$$\rho dxdyx^2 = \rho x^2 dxdy.$$

Thus the total moment of inertia is

$$I = \int_{x=0}^{a} \int_{y=0}^{b} \rho x^2 dxdy$$

$$= \rho b \int_{0}^{a} x^2 = \rho b \left| \frac{x^3}{3} \right|_0^a = \frac{1}{3} \rho b a^3$$

Since the total mass of rectangular plate is

$$M = \rho ab$$

the moment o inertia will be

$$I = \frac{1}{3} M a^2$$

Module No. 149

M.I. of Square Plate – Derivation

Problem Statement

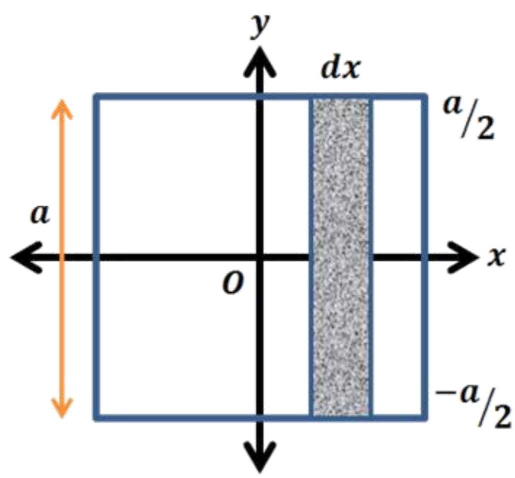
Calculate the moment of inertia of a uniform square plate with sides $\boldsymbol{a}$ about any axis through its center and lying in the plane of plate.

Solution

Consider a uniform square plate with length of its side to be a, we have to find out M.I. of this plate about any axis, through its center.
We consider this axis as $y - axis$.
Let ρ be the density of plate and the total area of square plate is a^2.
So density will be

$$\rho = \frac{M}{a^2}$$

We assume that the plate has been divided into vertical strips. Let us consider a strip from the whole square as shown in figure.



The strips are chosen in this way because each point on a particular strip is approximately the same distance from axis of rotation i.e. y-axis, the mass of the strip is δm and the width of each strip is δx, then the area of the strip will be $a\delta x$, so

$$\delta m = \rho a \delta x$$

Let the distance of the strip from y-axis is x, then the moment of inertia of strip will be

$$\delta I = \delta m x^2 = x^2 \rho a \delta x$$

So the moment of inertia of square is

$$I_{square} = \sum_{all\ strips} x^2 \rho a \delta x$$

$$I_{square} = \int_{-a/2}^{a/2} x^2 \rho a dx = \rho a \int_{-a/2}^{a/2} x^2 dx$$

$$= \rho a \left|\frac{x^3}{3}\right|_{-a/2}^{a/2} = \frac{\rho a}{3}\left|\frac{a^3}{8} - \frac{(-a^3)}{8}\right|$$

$$= \frac{\rho a}{3}\left(\frac{a^3}{4}\right) = \frac{\rho a^4}{12}$$

Now substitute $\rho = \frac{M}{a^2}$, we obtain

$$I_{square} = \frac{Ma^2}{12}$$

Module No. 150

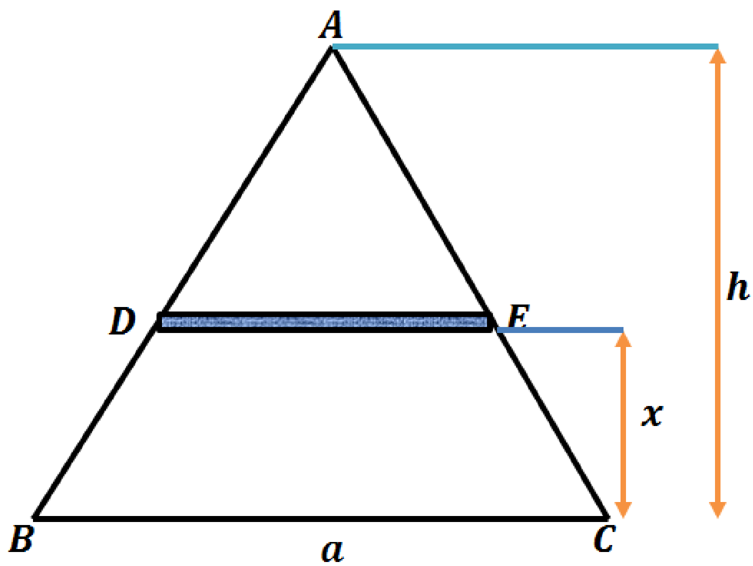
M.I. of Triangular Lamina – Derivation

Problem Statement

Find the moment of inertia of a uniform triangular lamina of mass M about one of its sides.

Solution

Let ABC be the lamina. We will find its M.I. about one of its side BC. We choose the X-axis and the origin as shown in figure.



At a distance x from BC we consider a strip of lamina of width dx. If the mass of the strip is dm then its moment of inertia about BC is = $x^2 dm$.
To find dm we note that the triangles ABC and ADE are similar. Therefore the ratios of sides are the same.
Hence

$$\frac{DE}{BC} = \frac{height\ of\ ADE}{height\ of\ ABC} = \frac{h-x}{h}$$

which gives

$$DE = \frac{h-x}{h} \times BC = \frac{h-x}{h} \times a$$

where h is the height of the triangle ABC at base BC and a is the length of BC. Now

$$\delta m = \rho(area) = \rho DE \delta x$$

$$= a\left(\frac{h-x}{h}\right)\delta x \rho$$

where ρ is the density.
Therefore the moment of inertia of triangular lamina about the side BC is

$$I = \int x^2 dm = \int_0^h a\left(\frac{h-x}{h}\right) x^2 dx\rho$$

$$= \frac{\rho a}{h}\int_0^h x^2(h-x)dx = \frac{\rho a}{h}\int_0^h (hx^2 - x^3)dx$$

$$= \frac{\rho a}{h}\left| h\frac{x^3}{3} - \frac{x^4}{4}\right|_0^h$$

$$= \frac{\rho a}{h}\left(\frac{h^4}{3} - \frac{h^4}{4}\right) = \frac{\rho a h^3}{12}$$

Now substitute density= $\rho = \frac{Mass}{Area} = \frac{M}{1/2ah}$
we obtain

$$I = \frac{1}{6}Mh^2$$

Hence the required expression for moment of inertia of the triangular lamina is $\frac{1}{6}Mh^2$.

Module No. 151

M.I. of Elliptical Plate along its Major Axis – Derivation

Problem Statement

Find the M.I. of a uniform elliptical plate with semi major axes and semi minor axes a, b respectively about its major axes.

Solution

We consider the elliptical plate as

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (1)

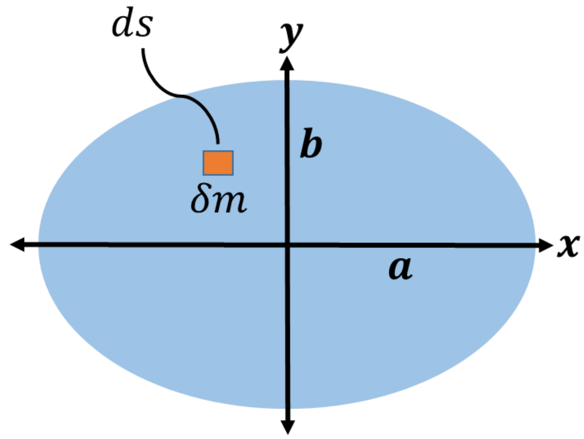
with semi major axes along x-axis as shown in figure.

From (1) we have

$$y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$$

So, let $y_1 = \frac{b}{a}\sqrt{a^2 - x^2}$,

We consider a small element of plate of mass δm of the elliptical plate will be ρds which will be equal to $\rho dxdy$. The moment of inertia if this element along x-axis will be equal to $I = \delta m y^2$.



Then the moment of inertia of whole plate will be

$$I_x = \int (\delta m) y^2 = \int_{plate} \rho ds\, y^2$$

$$= \rho \int_{-a}^{a} \left(\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} y^2 dy \right) dx$$

$$= \rho \int_{-a}^{a} \frac{2y_1^2}{3} dx = \frac{2\rho}{3} \int_{-a}^{a} \frac{b^3}{a^3} (a^2 - x^2)^{3/2} dx$$

due to symmetry, we can write it as

$$= \frac{4\rho b^3}{3a^3} \int_{0}^{a} (a^2 - x^2)^{3/2}\, dx$$

By making use of polar coordinates, substitute $x = a \sin\theta$, then $dx = a\cos\theta$ this integral becomes

$$I_x = \frac{4\rho b^3}{3a^3} \int_{0}^{\pi/2} a^3 \cos^3\theta\, (a\cos\theta) d\theta$$

$$= \frac{4\rho a b^3}{3} \int_{0}^{\pi/2} \cos^4\theta\, d\theta$$

$$= \frac{4\rho a b^3}{3} \frac{1 \times 3}{2 \times 4} \times \frac{\pi}{2}$$

$$= \frac{1}{4} \rho a b^3 \pi$$

$$= \frac{1}{4} a b^3 \pi \times \frac{M}{\pi a b} = \frac{1}{4} M b^2$$

where for elliptical plate, we have

$$\rho = \frac{M}{\pi a b}$$

is the required expression for M.I.

Module No. 152

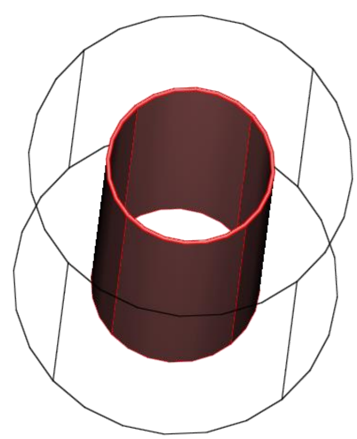
M.I. of a Solid Circular Cylinder - Derivation

Problem Statement

To find the moment of inertia of a solid circular cylinder of radius a, mass M and the height of the cylinder h about the axis of the cylinder.

Solution

Let's subdivide the solid circular cylinder into concentric cylindrical shells/ hollow cylinders, one of which is shown in the fig.



Let the mass of one shell isdm, height is same as h, thickness be dr and the radius be r then the density of the shell will be

$$\rho = \frac{dm}{2\pi r dr h}$$

or

$$dm = 2\pi \rho r dr h$$

Since we have

$$\rho = \frac{M}{V_{cylinder}} = \frac{M}{\pi a^2 h} = \frac{dm}{2\pi r dr h}$$

or

$$dm = \frac{2M}{a^2} r dr$$

Since the moment of inertia of the shell will be:

$$I_{shell} = r^2 dm$$

or

$$I_{shell} = r^2 \frac{2M}{a^2} r dr = \frac{2M}{a^2} r^3 dr$$

Thus the total moment of inertia of the solid circular cylinder will be

$$I_{cylinder} = \int_{r=0}^{a} I_{shell}$$

$$I_{cylinder} = \int_{r=0}^{a} \frac{2M}{a^2} r^3 dr$$

$$= \frac{2M}{a^2} \int_{r=0}^{a} r^3 dr$$

$$= \frac{2M}{a^2} \left| \frac{r^4}{4} \right|_0^a = \frac{2M}{a^2} \frac{a^4}{4}$$

$$I_{cylinder} = \frac{1}{2} M a^2$$

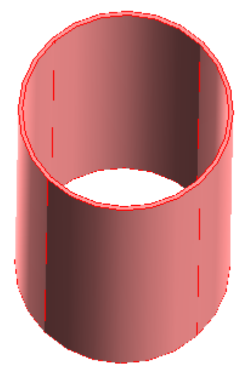
is the M.I of solid cylinder.

Module No. 153

M.I. of Hollow Cylindrical Shell - Derivation

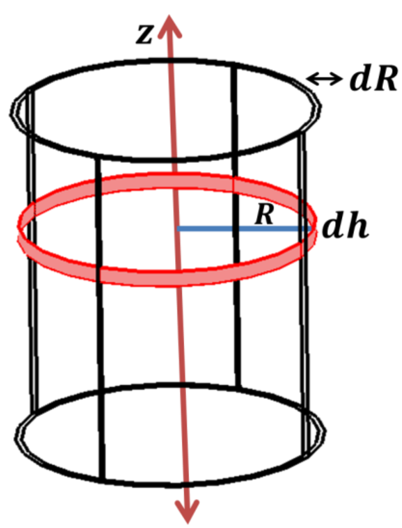
Problem

To find the moment of inertia of a hollow open cylindrical shell of radius R, thickness dR, mass M and height of shell h about the axis of shell.



Solution

Let's subdivide the hollow shell into small hoops/ rings, one of which is shown in the figure.



Let the mass of one ring isdm, height is dh, thickness be dr and the radius be R, then the mass of one ring will be

$$dm = \rho 2\pi R dR dh$$

Hence the moment of inertia of the ring of radius R will be

$$I_{ring} = dmR^2$$

or

$$I_{ring} = (\rho 2\pi R dR dh)R^2 = 2\rho\pi R^3 dR dh$$

In order to obtain the moment of inertia for the whole hollow cylindrical open shell, we will integrate

$$I = \int I_{ring} = \int_0^h 2\rho\pi R^3 dR dh$$

$$I = 2\rho\pi R^3 dR \int_0^h dh$$

$$= 2\rho\pi R^3 dR h$$

Since the density of the hollow cylindrical shell is

$$\rho = \frac{M}{2\pi R dR h}$$

Therefore

$$I = MR^2$$

is the M.I of hollow cylindrical open shell.

Module No. 154

M.I of Solid Sphere - Derivation

Problem

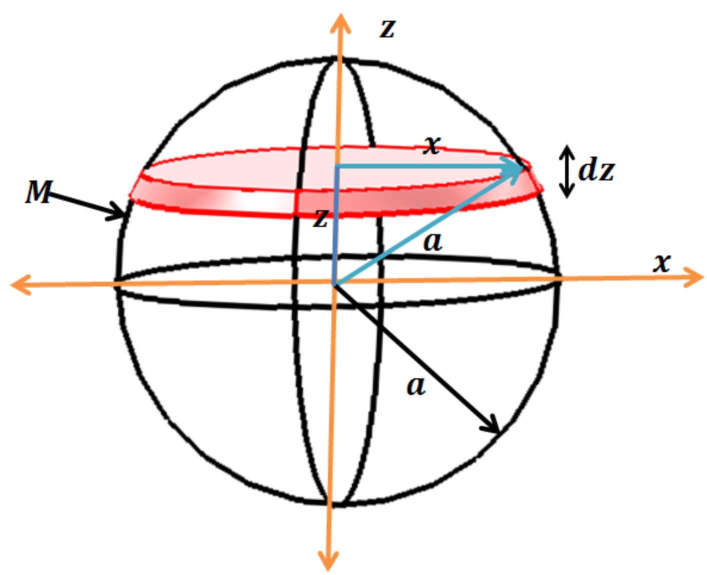
Find the moment of inertia of a uniform solid sphere of radius $\boldsymbol{a}$ and mass $\boldsymbol{M}$ about an axis (the z-axis) passing through the center.

Solution

For a uniform solid sphere, due to symmetry, we have

$I_{xx} = I_{yy} = I_{zz}$

In order to calculate the moment of Inertia of the sphere, we split the sphere into thin circular discs, one of which is shown in Figure.



We have already derived the expression for the moment of inertia of a representative disc of radius x, which is

$$I_{disk} = \frac{1}{2}x^2 dm$$

of an elementary disc of mass dm and the radius x.

As we know the mass = (density)(Area of disc)

therefore

$$dm = \rho\pi x^2$$

Hence moment of inertia of the sphere along z-axis will be

$$I_{zz} = \int_{-a}^{a} \frac{1}{2}\rho\pi x^4 dz$$

Now, to write "x"in

terms of z, we make

a triangle as shown in fig,

where

$$a^2 = x^2 + z^2, \quad \Longrightarrow x^2 = a^2 - z^2$$

$$I_{zz} = \int_{-a}^{a} \frac{1}{2}\rho\pi(a^2 - z^2)^2 dz$$

$= \frac{8}{15}\pi\rho a^5$

Since the mass of the sphere is

$$M = \frac{4}{3}\pi a^3 \rho$$

Therefore

$$I_{zz} = \frac{2}{5}Ma^2$$

Also

$$I_{xx} = I_{yy} = I_{zz} = \frac{2}{5}Ma^2$$

is the required moment of inertia of the sphere.

Module No. 155

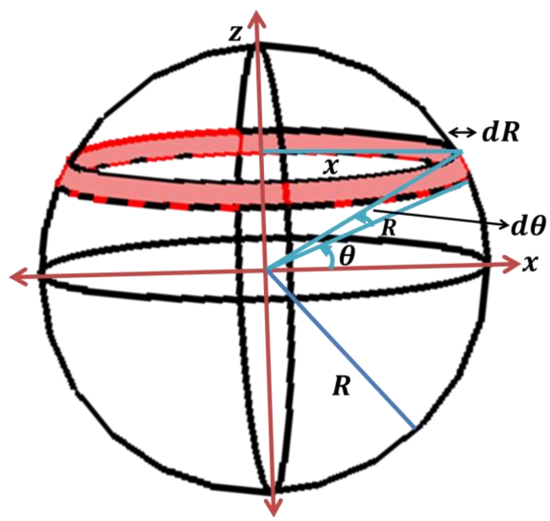
M.I. of the Hollow Sphere – Derivation

Problem

A thin uniform hollow sphere has a radius $\boldsymbol{R}$ and mass $\boldsymbol{M}$. Calculate its moment of inertia about any axis through its center.

Solution

In order to calculate the moment of inertia of the hollow sphere, we split the hollow sphere into thin hoops (rings), as shown in Figure.



We have already derived the expression for the moment of inertia of a representative hoop of radius x, which is

$$I = dmx^2$$

of an elementary ring of mass dm and the radius x.

The volume of the elementary ring is

$$dV = 2\pi x R d\theta dR$$

As we know the mass $=$ (density)(volume)

$$dm = \rho dV$$

therefore

$$dm = \rho 2\pi x R dR d\theta$$

Hence the moment of inertia of the small ring of radius x will be

$$I_{ring} = dm x^2$$

or

$$I_{ring} = (2\pi x \rho R dR d\theta)x^2 = 2\pi\rho R dR x^3 d\theta$$

which is the moment of inertia of ring of radius x chosen from the hollow sphere.

In order to obtain the moment of inertia for the whole hollow sphere, we will integrate

$$I = \int I_{ring} = \int_{-\pi/2}^{\pi/2} 2\pi\rho R dR x^3 d\theta$$

due to symmetry, we can write

$$I = 4\pi\rho R dR \int_{0}^{\pi/2} x^3 d\theta$$

To solve the integral, we need to write x in terms of θ. From fig we have

$$x = R\cos\theta$$

The integral becomes,

$$I = 4\pi\rho R dR \int_{0}^{\pi/2} (R\cos\theta)^3 d\theta$$

$$I = 4\pi\rho R^4 dR \int_{0}^{\pi/2} \cos^3\theta \, d\theta$$

$$I = 4\pi\rho R^4 dR \int_{0}^{\pi/2} \cos\theta\cos^2\theta \, d\theta$$

$$I = 4\pi\rho R^4 dR \int_0^{\pi/2} \cos\theta\,(1-\sin^2\theta)d\theta$$

$$I = 4\pi\rho R^4 dR \int_0^{\pi/2} \cos\theta\,(1-\sin^2\theta)d\theta$$

$$I = 4\pi\rho R^4 dR(\int_0^{\pi/2} \cos\theta\,d\theta - \int_0^{\pi/2} \sin^2\theta\cos\theta\,d\theta)$$

$$I = 4\pi\rho R^4 dR \left|\sin\theta - \frac{\sin^3\theta}{3}\right|_0^{\pi/2}$$

$$= 4\pi\rho R^4 dR \left(1 - \frac{1}{3}\right)$$

$$= 4\pi\rho R^4 dR \times \frac{2}{3}$$

$$= \frac{8}{3}\pi\rho R^4 dR$$

Since the density of the hollow sphere is

$$\rho = \frac{M}{V} = \frac{M}{4\pi R^2 dR}$$

Hence M.I of hollow sphere will be

$$I = \frac{2}{3}MR^2$$

Module No. 156

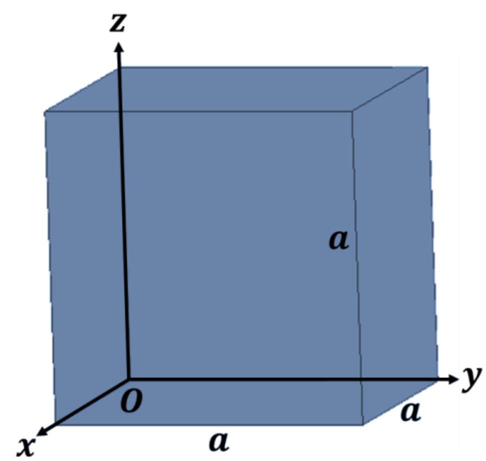
Inertia Matrix / Tensor of solid Cuboid

Problem Statement

Calculate the inertia matrix / inertia tensor of uniform solid cube at one of its corners.

Solution

Let the length of the edges be a of each side and let the axes be chosenalong the edges as shown in the figure.



By definition

$$I_{11} \equiv I_{xx} = \int \rho(r)(y^2 + z^2)dV$$

Since the box is made of uniform material, the density ρ must be constant. Therefore

$$I_{xx} = \rho \int_0^a \int_0^a \int_0^a (y^2 + z^2)dxdydz$$

$$= \rho \int_0^a \int_0^a (y^2 + z^2)dydz \int_0^a dx$$

$$= \rho a \int_0^a \int_0^a (y^2 + z^2)dydz$$

$$= \rho a\left[\int_0^a\int_0^a y^2 dydz + \int_0^a\int_0^a z^2 dydz\right]$$

$$= \rho\left(\frac{a^4}{3} + \frac{a^4}{3}\right)$$

$$I_{xx} = \rho a\left(\frac{2a^4}{3}\right) = \rho\left(\frac{2a^5}{3}\right)$$

Since we know that for the cube

$$\rho = \frac{M}{a^3}$$

We obtain

$$I_{xx} = \frac{2M}{3}a^2$$

Similarly, due to symmetry, we can write

$$I_{yy} = \frac{2M}{3}a^2$$

$$I_{zz} = \frac{2M}{3}a^2$$

Now for the product of inertia, we have

$$I_{12} = -\int \rho xydV$$

$$= -\rho\int_0^a\int_0^a\int_0^a xydxdydz$$

$$= -\rho\int_0^a xdx\int_0^a ydy\int_0^a dz$$

$$= -\rho a\frac{a^2}{2}\frac{a^2}{2} = -\rho\frac{a^5}{4}$$

Again using

$$\rho = \frac{M}{a^3}$$

We obtain

$$I_{12} = -\frac{Ma^2}{4}$$

Similarly,

$$I_{12} = -\frac{Ma^2}{4}$$

$$I_{12} = -\frac{Ma^2}{4}$$

The required inertia matrix / inertia tensor will be

$$I_{ij} = \begin{bmatrix} \frac{2M}{3}a^2 & -\frac{Ma^2}{4} & -\frac{Ma^2}{4} \\ -\frac{Ma^2}{4} & \frac{2M}{3}a^2 & -\frac{Ma^2}{4} \\ -\frac{Ma^2}{4} & -\frac{Ma^2}{4} & \frac{2M}{3}a^2 \end{bmatrix}$$

Module No. 157

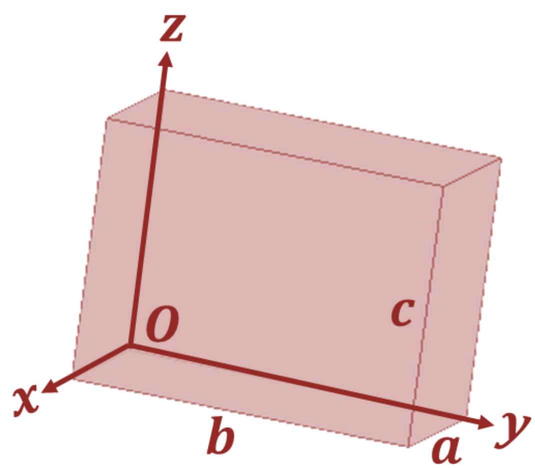
Inertia Matrix / Tensor of solid Cuboid

Problem Statement

Calculate the inertia matrix of uniform solid cuboid (parallelepiped) at one of its corner.

Solution

Let the length of the edges be a, b, c and let the axes be chosen along the edges as shown in the figure.



By definition

$$I_{11} \equiv I_{xx}$$

$$= \int \rho(r)(y^2 + z^2)dV$$

Since the box is made of uniform material, the density ρ must be constant. Therefore

$$I_{xx} = \rho \int_0^a \int_0^b \int_0^c (y^2 + z^2)dxdydz$$

$$= \rho \int_0^b \int_0^c (y^2 + z^2)dydz \int_0^a dx$$

$$= \rho a \int_0^b \int_0^c (y^2 + z^2) dydz$$

$$= \rho a \left[\int_0^b \int_0^c y^2 dydz + \int_0^b \int_0^c z^2 dydz \right]$$

$$= \rho a \left(c \frac{b^3}{3} + b \frac{c^3}{3} \right)$$

$$I_{xx} = \rho \frac{abc}{3} (b^2 + c^2)$$

using relation

$$\rho = \frac{M}{abc}$$

$$I_{xx} = \frac{M}{3} (b^2 + c^2)$$

Similarly, due to symmetry, we can write

$$I_{yy} = \frac{M}{3} (a^2 + c^2)$$

$$I_{zz} = \frac{M}{3} (a^2 + b^2)$$

Now for the product of inertia, we have

$$I_{12} = -\int \rho xydV$$

$$= -\rho \int_0^a \int_0^b \int_0^c xydxdydz$$

$$= -\rho \int_0^a xdx \int_0^b ydy \int_0^c dz$$

$$-\rho c \frac{a^2}{2} \frac{b^2}{2} = -\rho \frac{a^2 b^2 c}{4}$$

Again using

$$\rho = \frac{M}{abc}$$

we get

$$I_{12} = -\frac{Mab}{4}$$

Similarly,

$$I_{23} = -\frac{Mbc}{4}$$

$$I_{31} = -\frac{Mac}{4}$$

The required inertia matrix / inertia tensor will be

$$I_{ij} =$$

$$\begin{bmatrix} \frac{M}{3}(b^2 + c^2) & -\frac{Mab}{4} & -\frac{Mac}{4} \\ -\frac{Mab}{4} & \frac{M}{3}(a^2 + c^2) & -\frac{Mbc}{4} \\ -\frac{Mac}{4} & -\frac{Mbc}{4} & \frac{M}{3}(a^2 + b^2) \end{bmatrix}$$

Module No. 158

M.I of Hemi-Sphere – Derivation

Problem Statement

Find the moment of inertia of a uniform hemisphere of radius a about its axis of symmetry.

Solution

We will use the spherical polar coordinates (r, θ, φ). Their use makes computational work simpler.

Their range of variation for hemisphere will be

$$0 \leq r < a$$

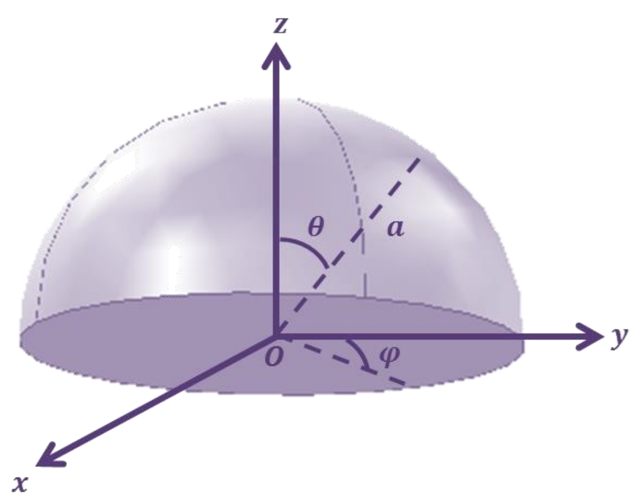
$$0 \leq \theta \leq {}^{\pi}/_{2}$$

$$0 \leq \varphi \leq 2\pi$$

$$x = r\sin\theta\cos\varphi\,,$$

$$y = r\sin\theta\sin\varphi\,, \;\; z = r\cos\theta$$

We choose the z-axis as the axis of symmetry.



Hence

$$I_{zz} = \int \rho(r)(x^2 + y^2)dV$$

$$= \rho \int (x^2 + y^2)dV$$

Now calculate $x^2 + y^2$ in terms of sp. coordinates

$$x^2 + y^2 = r^2(\sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi)$$

$$= r^2\sin^2\theta\,(\cos^2\varphi + \sin^2\varphi)$$

$$= r^2\sin^2\theta$$

and the element of volume in spherical polar coordinates is given by

$$dV = dr(rd\theta)(r\sin\theta d\varphi) = r^2\sin\theta\, drd\theta d\varphi$$

Therefore

$$I_{zz} = \rho \int_0^a \int_0^{\pi/2} \int_0^{2\pi} r^4 \sin^3\theta\, drd\theta d\varphi$$

$$= \int_0^a r^4 dr \int_0^{\pi/2} \sin^3\theta\, d\theta \int_0^{2\pi} d\varphi$$

$$= 2\pi\rho\frac{a^5}{5}\int_0^{\pi/2} \sin^3\theta\, d\theta$$

$$= 2\pi\rho\frac{a^5}{5}\int_0^{\pi/2} \frac{3\sin\theta - \sin 3\theta}{4}d\theta$$

$$= (2\pi\rho\frac{a^5}{5})\frac{1}{4}\left|-3\cos\theta + \frac{\cos 3\theta}{3}\right|_0^{\pi/2}$$

$$= \pi\rho\frac{a^5}{10}[(0) - (-3 + 1/3)]$$

$$= \pi\rho\frac{a^5}{10}(\frac{8}{3})$$

$$= 4\pi\rho\frac{a^5}{15}$$

By using the relation for density

$$\rho = \frac{M}{{}^2\!/_3\,\pi a^3}$$

we obtain

$$I_{zz} = \frac{2}{5}Ma^2$$

which is required expression for moment of inertia of hemisphere.

Module No. 159

M.I. of Ellipsoid –Derivation

Problem

Find the moment of inertia and product of inertia for the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

w.r.to its axes of symmetry.

Solution

By definition

$$I_{11} \equiv I_{xx}$$

$$= \int_{ellipsoid} \rho(r)(y^2+z^2)dV$$

If we put

$$\frac{x}{a}=x',\ \frac{y}{b}=y',\ \frac{z}{c}=z'$$

then

$$x=ax',\ y=by',\ z=cz'$$

$$\Longrightarrow dx=adx',\ \ dy=bdy',\ \ dz=cdz'$$

and

$$dV=dxdydz=abcdx'dy'dz'$$

Now under the above transformation, the given ellipsoid is transformed into the unit sphere S:

$$x'^2+y'^2+z'^2=1$$

The integration is now over the region enclosed by the unit sphere.

$$I_{11} = \rho \int_R (b^2 y'^2 + c^2 z'^2) abcdx'dy'dz'$$

or

$$I_{11} = \rho ab^3 c \int_R y'^2 dV' + \rho abc^3 \int_R z'^2 dV'$$

where $dV' = dx'dy'dz'$

Now because of symmetry

$$\int_R y'^2\, dV' = \int_R z'^2\, dV'$$

Now we solve one integral.

We use the spherical polar coordinates (r, θ, φ).

Their range of variation will be

$$0 \le r < 1,\ \ 0 \le \theta \le \pi,\ \ 0 \le \varphi \le 2\pi$$

$$x = r \sin\theta \cos\varphi\,,\quad y = r \sin\theta \sin\varphi\,,\ z = r\cos\theta$$

and

$$dV' = dr(rd\theta)(r \sin\theta d\varphi) = r^2 \sin\theta\, drd\theta d\varphi$$

Thus

$$\int_S y'^2\, dV' = \int_0^1 \int_0^\pi \int_0^{2\pi} r^4 \sin^3\theta \sin^2\varphi\, drd\theta d\varphi$$

$$= \int_0^1 r^4 dr \int_0^\pi \sin^3\theta\, d\theta \int_0^{2\pi} \sin^2\varphi\, d\varphi$$

$$= \frac{1}{5}\frac{1}{4}\left|-3\cos\theta + \frac{\cos 3\theta}{3}\right|_0^\pi \frac{1}{2}\left|\varphi - \frac{\sin 2\varphi}{2}\right|_0^{2\pi}$$

Thus

$$\int_S y'^2 \, dV' = \frac{2\pi}{15}$$

Therefore on substitution

$$I_{11} = \rho a b^3 c \frac{2\pi}{15} + \rho a b c^3 \frac{2\pi}{15}$$

$$I_{11} = \rho abc \frac{2\pi}{15} (b^2 + c^2)$$

$$= \frac{M}{(4/3)\pi abc} abc \times \frac{2\pi}{15} (b^2 + c^2)$$

$$I_{11} = \frac{M}{10} (b^2 + c^2)$$

Similarly,

$$I_{22} = \frac{M}{10} (c^2 + a^2)$$

$$I_{33} = \frac{M}{10} (a^2 + b^2)$$

For product of inertia

$$I_{12} \equiv I_{xy} = - \int_{ellipsoid} xydV$$

$$= -\rho \int_R ax'by'(abcdx'dy'dz')$$

$$= -\rho a^2 b^2 c \iiint_R x'y'dx'dy'dz'$$

Using the polar coordinates $(r,\, \theta,\, \varphi)$

$$I_{12} = -\rho a^2 b^2 c \int_0^{2\pi} \int_0^{\pi} \int_0^{1} r \sin\theta \cos\varphi (\, \mathrm{r} \sin\theta \sin\varphi) r^2 \sin\theta \, dr d\theta d\varphi$$

$$= 0$$

Similarly, we can obtain

$$I_{23} = 0$$

$$I_{31} = 0$$

Module No. 160

Example 1 of Moment of Inertia

Problem

Particles of masses $2m$, $3m$ and $4m$ are held in a rigid light framework at points $(0,1,1)$, $(1,1,0)$ and $(-1,\ 0,\ 1)$ resp.

Show that the M. I. of the system about the $x, y, z-$ axis are 11, $13,\ 12$ respectively.

Solution

Since the masses are 2, 3 and 4 i.e. $m_1 = 2,\ m_2 = 3$ and $m_3 = 4$, and the points are given by $P_1(0,1,1), P_2(1,1,0)$ and $P_3(-1,0,1)$.

We are required to find out the moment of inertia about x-axis, y-axis and z-axis i.e. I_{xx}, I_{yy}, I_{zz}.

$$I_{xx} = \sum_i m_i(\,y_i^2 + z_i^2)$$

$$= m_1(y_1^2 + z_1^2) + m_2(y_2^2 + z_2^2) + m_3(y_3^2 + z_3^2)$$

$$= 2(1+1) + 3(1+0) + 4(1+0)$$

$$= 4 + 3 + 4 = 11$$

So,

$$I_{xx} = 11$$

$$I_{yy} = \sum_i m_i(\,x_i^2 + z_i^2)$$

$$= m_1(x_1^2 + z_1^2) + m_2(x_2^2 + z_2^2) + m_3(x_3^2 + z_3^2)$$

$$= 2(0+1) + 3(1+0) + 4(1+1)$$

$$= 2 + 3 + 8 = 1$$

So,

$$I_{yy} = 13$$

$$I_{zz} = \sum_i m_i(x_i^2 + y_i^2)$$

$$= m_1(y_1^2 + x_1^2) + m_2(x_2^2 + y_2^2) + m_3(x_3^2 + y_3^2)$$

$$= 2(0 + 1) + 3(1 + 1) + 4(1 + 0)$$

$$= 2 + 6 + 4 = 12$$

So,

$$I_{zz} = 12$$

- Hence Showed

Module No. 161

Example 2 of Moment of Inertia

Problem Statement

A boy of mass $M = 30\text{kg}$ is running with a velocity of $3\ \mathrm{m/sec}$ on ground just tangentially to a merry-go-round which is at rest. The boy suddenly jumps on the merry-go-round. Calculate the angular velocity acquired by the system. The merry-go-round has a radius of $r = 2\text{m}$ and a mass $m = 120\text{kg}$ and its moment of inertia is $120\ \text{kgm}^{-2}$.

Solution

The merry-go-round rotates about an axis which we regard as passing through its c.m.

Let's give the following notations:

- M.I of boy $= I_1$
- M.I of merry go round $= I_2$
- Vel. of boy $= v_1$
- Vel. of merry go round $= v_2$
- Angular vel. of boy $= \omega_1$
- Angular vel. of merry go round $= \omega_2$

The moment of inertia I_1 of the boy about the axis of rotation can be found by

$$I_1 = Md^2 = 30 \times 2^2 = 120kgm^{-2}$$

The moment of inertia I_2 of merry go round is given to be

$$I_2 = 120kgm^{-2}$$

Since the velocity of the boy about the merry-go-round is $v_1 = 3\text{ms}^{-1}$, therefore his angular velocity ω_1 about the axis of rotation is therefore

$$\omega_1 = \frac{v}{d} = \frac{3}{2} = 1.5 \text{ radian per second}$$

Initially the merry go round is at rest, therefore $v_2 = 0$, and thus its angular velocity $\omega_2 = 0$.
We ignore friction and therefore there is no external torque on the system.
Hence by the law of conservation of angular momentum

$$(I_1 + I_2)\omega = I_1\omega_1 + I_2\omega_2$$

$$\omega = \frac{I_1\omega_1 + I_2\omega_2}{(I_1 + I_2)}$$

where ω is the angular velocity of the system when the boy jumps on the merry-go-round.

$$\omega = \frac{120(5) + 0}{120 + 120}$$

$$= 0.75 \text{ radian per sec}$$

Module No. 162

Example 3 of Moment of Inertia

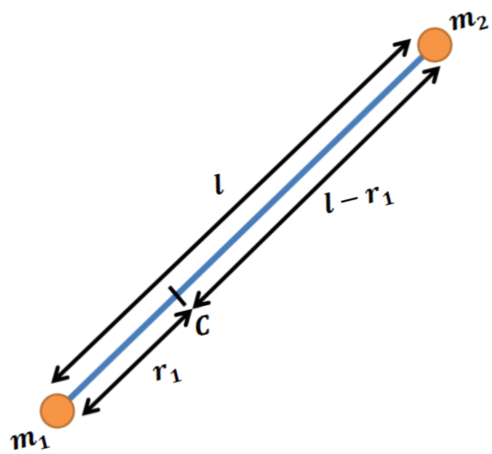
Problem Statement[1]

Two particles of masses m_1 and m_2 are connected by a rigid massless rod of length l and moves freely in a plane. Show that the M. I. of the system about an axis perpendicular to the plane and passing through the center of mass is Ml^2 where

$$M = \frac{m_1 m_2}{m_1 + m_2}$$

Solution

Let r_1 be the distance of mass m_1 from center of mass C. Then $l - r_1$ is the distance of the mass m_2 from C.Since C is the center of the mass.



So,

$$m_1 r_1 = m_2(l - r_1)$$

$$\Rightarrow m_1 r_1 = m_2 l - m_2 r_1$$

$$\Rightarrow (m_1 + m_2) r_1 = m_2 l$$

$$\Rightarrow r_1 = \frac{m_2 l}{(m_1 + m_2)}$$

$$\Longrightarrow l - r_1 = l - \frac{m_2 l}{(m_1 + m_2)}$$

$$\Longrightarrow l - r_1 = \frac{m_1 l + m_2 l - m_2 l}{(m_1 + m_2)}$$

$$\Longrightarrow l - r_1 = \frac{m_1 l}{(m_1 + m_2)}$$

Thus the M.I. about an axis through C is

$$m_1 r_1^2 + m_2 (l - r_1^2)^2$$

$$= m_1 \left(\frac{m_2 l}{m_1 + m_2}\right)^2 + m_2 \left(\frac{m_1 l}{m_1 + m_2}\right)^2$$

$$= \frac{m_1 m_2 l^2}{(m_1 + m_2)^2}(m_1 + m_2) = \frac{m_1 m_2}{m_1 + m_2} l^2$$

$$I = Ml^2$$

where

$$M = \frac{m_1 m_2}{m_1 + m_2}$$

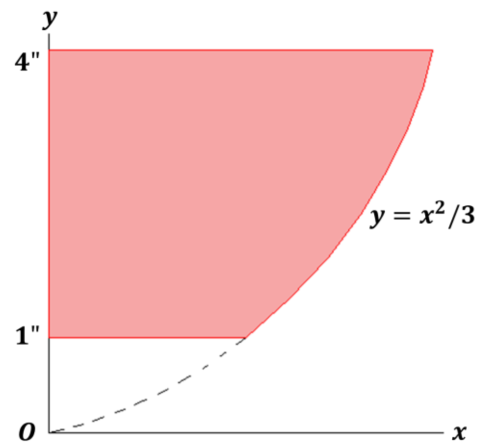
Hence showed.

Module No. 163

Example 4 of Moment of Inertia

Problem (© engineering.unl.edu)

Calculate the moment of inertia of the shaded area given in figure about y-axis.



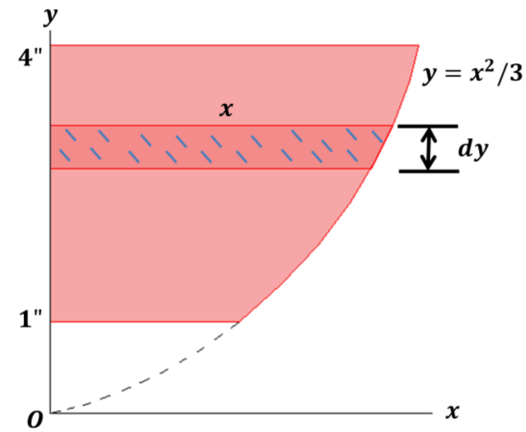
Solution

Here, we have

$y = x^2/3$ (1)

We consider an elementary strip from the shaded area whose M.I is

$$I_{strip} = \frac{1}{3}x^2(xdy) = \frac{1}{3}x^3dy$$



Then by integration, we obtain the moment of inertia of the whole shaded area

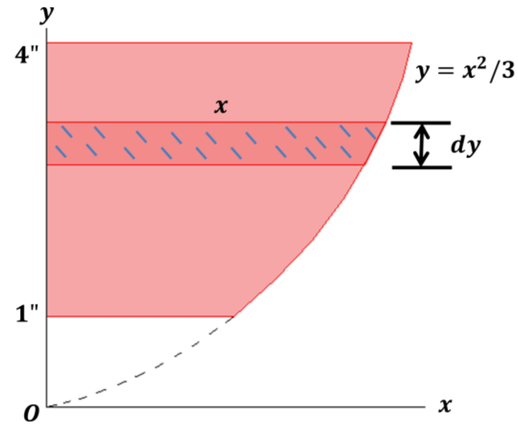
$$I_y = \sqrt{3}\int_1^4 y^{3/2}dy = \sqrt{3}\frac{2}{5}\left|y^{5/2}\right|_1^4$$

$$= \sqrt{3}\frac{2}{5}\left[(4)^{5/2} - (1)^{5/2}\right]$$

$$= \frac{2\sqrt{3}}{5}\left[(4)^{5/2} - (1)^{5/2}\right]$$

$$= \frac{2\sqrt{3}}{5}(32 - 1)$$

$$= 21.5 \text{ inch}^4$$



By using (1), we get

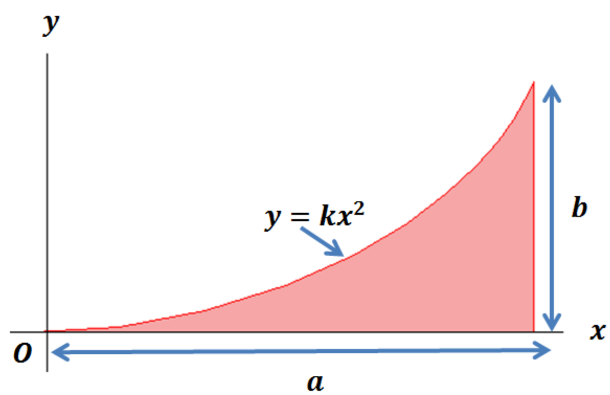
$$I_{strip} = \sqrt{3}y^{3/2}dy$$

Module No. 164

Example 5 of Moment of Inertia

Problem

Determine the moment of inertia of the shaded area shown in figure with respect to each of the coordinate axes.



Solution

Here we have

$y = kx^2$ (1)

From fig. we have

$x = a,\ y = b,$ then

$$b = ka^2 \Longrightarrow k = \frac{b}{a^2}$$

Substituting the value

of k in (1), we obtain

$$y = \frac{b}{a^2}x^2 \text{ or } x = \frac{a}{\sqrt{b}}\sqrt{y}$$

Now the moment of inertia along x-axis will be

$$I_x = \int_A y^2 dA = \int_0^b y^2 (a - x) dy$$

$$= \int_0^b y^2 \left(a - \frac{a}{\sqrt{b}} \sqrt{y} \right) dy$$

$$= a \int_0^b y^2 dy - \frac{a}{\sqrt{b}} \int_0^b y^{5/2} dy$$

$$= a \left| \frac{y^3}{3} \right|_0^b - \frac{a}{\sqrt{b}} \left| \frac{2}{7} y^{7/2} \right|_0^b$$

$$= \frac{ab^3}{3} - \frac{a}{\sqrt{b}} \frac{2}{7} b^{7/2}$$

$$= \frac{ab^3}{3} - \frac{2ab^3}{7}$$

$$I_x = \frac{ab^3}{21}$$

$$I_y = \int_A x^2 dA = \int_0^a x^2 y dx$$

$$= \frac{b}{a^2} \left| \frac{x^5}{5} \right|_0^a = \frac{b}{a^2} \frac{a^5}{5}$$

$$I_y = \frac{a^3 b}{5}$$

Hence

$$I_x = \frac{ab^3}{21}$$

and

$$I_y = \frac{a^3 b}{5}$$

are moment of inertia about x-axis and y-axis respectively.

Module No. 165

Example 5 of Moment of Inertia

Theorem Statement

The moment of inertia of a rigid body about a given axis is equal to the same about a parallel axis through the centroid plus the moment of inertia due to the total mass placed at the centroid, (the last quantity will be referred to as moment of inertia of the centroid).

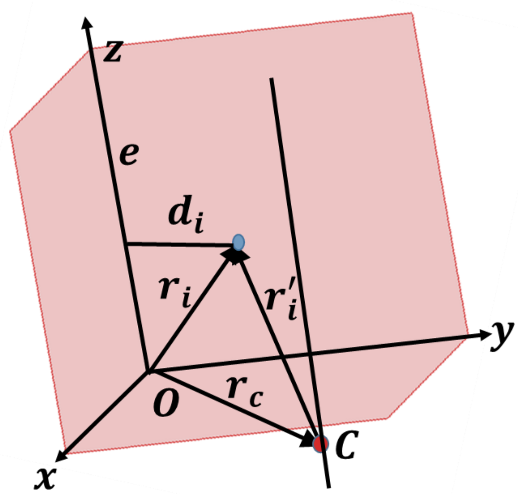
Proof

By definition, if I denotes the M.I about the given axis, and d_i denotes the distance of i^{th} particle from the given axis, then

$$I = \sum_i m_i d_i^2$$

If e is the unit vector in the direction of the axis, then we have the following result

$$d_i{}^2 = (e \times r_i)^2$$



Let r_c denotes the position vector of the centroid and

r_i' the position vector of the i^{th} particle w.r.t to the centroid, then

$$r_i = r_c + r_i'$$

On making substitutions, we obtain

$$I = \sum_i m_i \left[e \times (r_c + r_i')\right]^2$$

$$= \sum_i m_i \left[e \times r_c + e \times r_i'\right]^2$$

$$= \sum_i m_i \left[(e \times r_c)^2 + (e \times r_i')^2 + 2\left(e \times r_c\right).(e \times r_i')\right]$$

$$= \sum_i m_i (e \times r_c)^2 + \sum_i m_i (e \times r_i')^2$$

$$+2\left(e \times r_c\right).\sum_i m_i (e \times r_i')$$

$$= \sum_i m_i {d_c}^2 + \sum_i m_i d_i'^2 + 2\left(e \times r_c\right).e \times \sum_i m_i r_i'$$

But $\sum_i m_i r_i' = 0$ (We studied in earlier module).

Therefore

$$I = (\sum_i m_i){d_c}^2 + \sum_i m_i d_i'^2$$

$$= M{d_c}^2 + I'$$

where

$$M = \sum_i m_i$$

is the total mass of the system.

or

$$I = I_0 + I'$$

where I_0 denotes the moment of inertia of the centroid, and I' denotes the M.I of the system w.r.to a parallel axis through the centroid.

Module No. 166

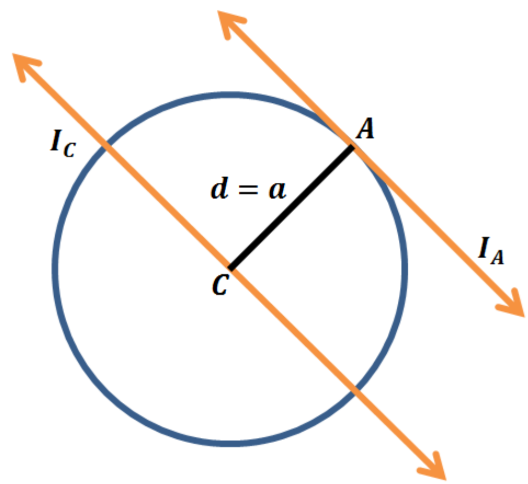
Example 1 of Parallel Axis Theorem

Problem

Use the parallel axis theorem to find the moment of inertia of a solid circular cylinder about a line on the surface of the cylinder and parallel to axis of cylinder.

Solution

Suppose the cross section of cylinder as in figure. Then the axis of the cylinder is passing through the point C, while the line on the surface of cylinder is passing through A. So, we have to find out M.I of circular cylinder about a line passing through the point A whose radius is a (radius of circular cylinder) and mass is M.



By parallel axis theorem

$$I_A = I_c + Ma^2 \qquad (1)$$

Since I_C which is the moment of inertia of a solid circular cylinder about an axis passing from the center of mass is defined by

$$I_C = \frac{1}{2} Ma^2 \qquad (2)$$

where a is the radius of a solid circular cylinder.

By substituting equation (2) in equation (1) we have

$$I_A = \frac{1}{2} Ma^2 + Ma^2$$

$\Longrightarrow$

$I_A = (\frac{1}{2} + 1)\ Ma^2$

$$I_A = \frac{3}{2} Ma^2$$

Module No. 167

Example 2 of Parallel Axis Theorem

Problem

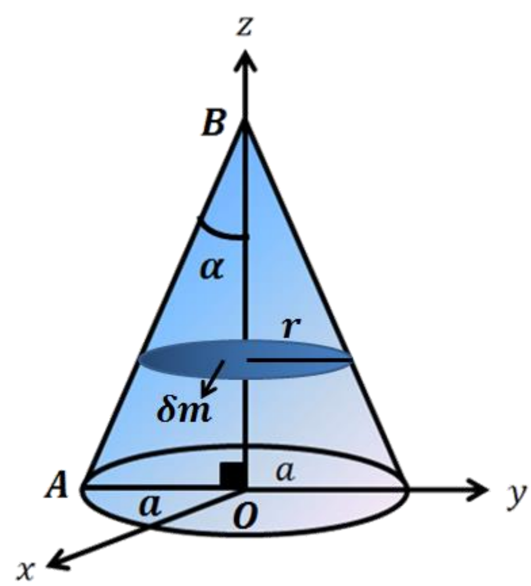
Prove that the moment of inertia of a uniform right circular cone using parallel axis theorem of mass m, height h and semi vertical angle α about a diameter of its base is

$$Mh^2(3\tan^2\alpha + 2)/20$$

Solution

In the case of M.I about its diameter, we consider the elementary disc of mass δm whose moment of inertia about a diameter will be

$$\delta I_0 = \frac{1}{4}r^2\delta m$$



We note that the diameter passes through the center (which is also the centroid) of the elementary disc.

Hence by parallel axis theorem, the M.I. δI of the elementary disc about a parallel axis (parallel diameter) at the base is given by

$$\delta I = \delta I_0 + (\delta m) z^2$$

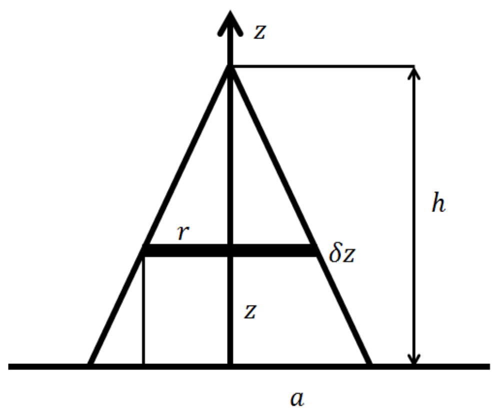
$$= \frac{1}{4} r^2 \delta m + \delta m z^2 = \delta m (\frac{1}{4} r^2 + z^2)$$

$$= \rho \pi r^2 \delta z (\frac{1}{4} r^2 + z^2) = \rho \pi \left(\frac{1}{4} r^4 + r^2 z^2 \right) \delta z$$

From the similar triangles,

we have

$$\frac{r}{a} = \frac{h - z}{h} \quad \text{or} \quad r = a \frac{h - z}{h}$$



Therefore $= \rho \pi \left[\frac{a^4}{4h^4} (h - z)^4 + \frac{a^2}{h^2} (h - z)^2 z^2 \right] \delta z$

$$= \rho \pi \left[\frac{a^4}{4h^4} (h - z)^4 + \frac{a^2}{h^2} (h^2 z^2 - 2hz^3 + z^4) \right] \delta z$$

Therefore M.I of complete right circular cone about a diameter is given by

$$I = \rho \pi \int_0^h \left\{ \frac{a^4}{4h^4} (h - z)^4 + \frac{a^2}{h^2} (h^2 z^2 - 2hz^3 + z^4) \right\} \delta z$$

$$I = \rho \pi \left(\frac{a^4}{4h^4} \frac{h^5}{5} + \frac{a^2}{h^2} \frac{h^5}{30} \right)$$

Since we know that $\rho = \frac{M}{(1/3) \pi a^2 h}$

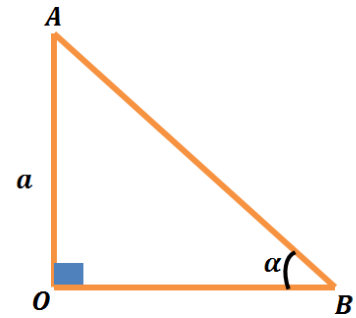
$$I = \frac{M}{20} (3a^2 + 2h^2)$$

Since the semi vertical angle of

the right circular cone is α,

So by right triangle AOB, we have

$$\tan\alpha = \frac{AO}{OB} = \frac{a}{h}$$



$a = h\tan\alpha$

Therefore

$$I = \frac{M}{20}[3(\mathrm{h}\tan\alpha)^2 + 2h^2]$$

or

$$I = \frac{Mh^2}{20}[3\tan^2\alpha + 2]$$

Hence proved.

Module No. 168

Example 3 of Parallel Axis Theorem

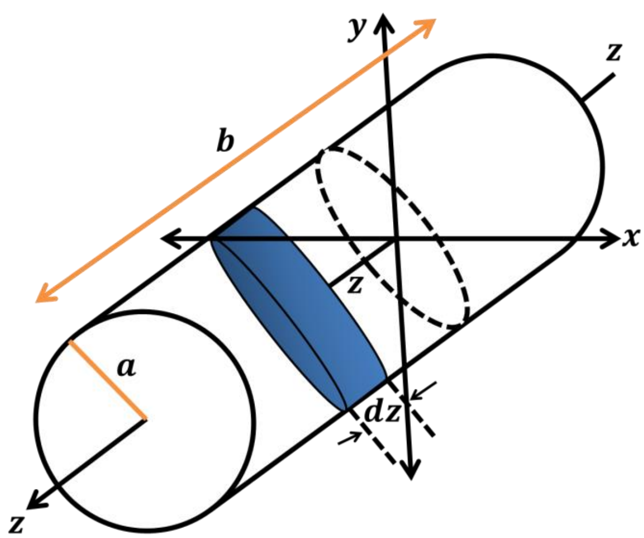
Problem[1]

Find the moment of inertia of a uniform circular cylinder of length b and radius a about an axis through the center and perpendicular to the central axis, namely I_x or I_y.

Solution

Consider the elementary disc of mass dm and thickness dz located at a distance z from the xy plane. Then the moment of inertia about a diameter will be

$$\delta I_0 = \frac{1}{4}a^2 dm$$



Then, using the parallel-axis theorem, moment of inertia of the thin disc about the $x - axis$ will be

$$dI_x = \frac{1}{4}a^2 dm + z^2 dm$$

where $dm = \rho\pi a^2 dz$.

Thus

$$I_x = \rho \pi a^2 \int_{-b/2}^{b/2} \left(\frac{1}{4}a^2 + z^2\right) dz$$

$$= \rho \pi a^2 \left(\frac{1}{4}a^2 b + \frac{1}{12}b^3\right)$$

but the mass of the cylinder m is $\rho \pi a^2 b$,

therefore

$$I_x = m\left(\frac{1}{4}a^2 + \frac{1}{12}b^2\right)$$

Due to symmetry we have

$$I_x = I_y = m\left(\frac{1}{4}a^2 + \frac{1}{12}b^2\right)$$

Module No. 169

Example 4 of Parallel Axis Theorem

Problem

Calculate the moment of inertia I_{cm} for a uniform rod of length l and mass M rotating about an axis through the center, perpendicular to the rod.

Solution

In order to calculate the moment of inertia through the center of mass c.m., we use parallel axes theorem.

In a transparent notation

$$I_l = I_{cm} + Md^2 \tag{1}$$

where d is the distance between the origin and the center of mass and $d = {}^{l}/_{2}$.

Also I_l is the moment of inertia of rod about one of its end (which we calculated earlier).

Here

$$I_l = \frac{1}{3}Ml^2 \tag{2}$$

From (1), we have

$$I_{cm} = I_l - Md^2$$

Substituting value from (2) in (1)

$$I_{cm} = \frac{1}{3}Ml^2 - M\left(\frac{l}{2}\right)^2 = \frac{1}{3}Ml^2 - \frac{1}{4}Ml^2$$

$$I_{cm} = \frac{1}{12}Ml^2$$

which is required moment of inertia about its center of mass.

Module No. 170

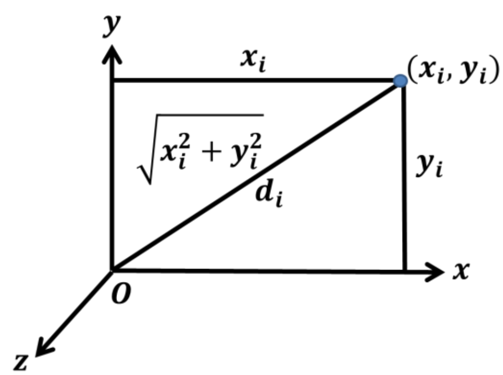
Perpendicular Axis Theorem

Theorem Statement

The moment of inertia of a **plane rigid body** about an axis perpendicular to the body is equal to the sum of the moment of inertia about two mutually perpendicular axis lying in the plane of the body and meeting at the common point with the given axis.

Proof

We choose the coordinates such that XY axes lie in the plane of the rigid body and the Z-axis is perpendicular to it.



Then the theorem can be stated as

$$I_{33} = I_{11} + I_{22}$$

$$or$$

$$I_{zz} = I_{xx} + I_{yy}$$

where $I_{11} = I_{xx}$ etc. are M.I about the x-axis etc.

Since the z-axis has been

chosen to be perpendicular

to the laminar body, therefore

$$I_{33} = I_{zz} = \sum_i m_i d_i^2$$

where d_i is the distance of the i^{th} particle (lying in the xy-plane) from the z-axis.

If we denote the coordinate of this particle by (x_i, y_i), then

$$d_i^2 = x_i^2 + y_i^2$$

and therefore

$$I_{33} = I_{zz} = \sum_i m_i(x_i^2 + y_i^2)$$

$$= \sum_i m_i x_i^2 + \sum_i m_i y_i^2$$

$$= I_{xx} + I_{yy}$$

- Hence the proof.

Module No. 171

Example 1 of Perpendicular Axis Theorem

Problem Statement

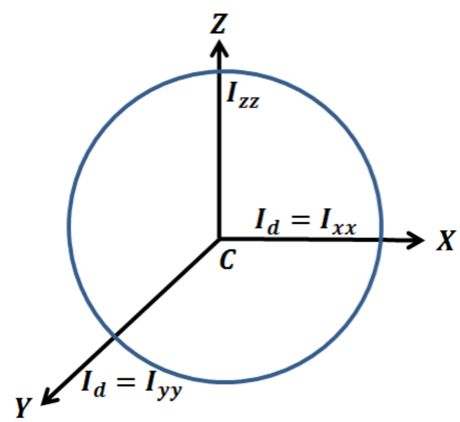
Find the moment of inertia of a uniform circular plate (or disc) about any diameter.

Proof

Since we have deduced that moment of inertia of a uniform circular disc is

$I_{disc} = \frac{1}{2}\,Ma^2$

which is about a line passing through center and perpendicular to the plane.



Considering the same axis with same terms in 3-dim body (i.e. $z - axis$ passing through center of mass and perpendicular to $xy - plane$), we have using perpendicular axis theorem

$$I_{zz} = I_{xx} + I_{yy} \qquad (1)$$

We have to find out M.I. of uniform circular plate (disc) about any of its diameters which are along $x - axis$ and $y - axis$ (i.e. we have to find out I_{xx} or I_{yy} both of them are equal to I_d)

Since

$I_{xx} = I_d = I_{yy}$

So equation (1) becomes

$$I_{zz} = I_d + I_d$$

or

$I_{zz} = 2I_d$

$$I_d = \frac{1}{2} I_{zz}$$

$$I_d = \frac{1}{2}(\frac{1}{2}\ \mathrm{M}a^2)$$

($\therefore I_{zz} = I_{disc} = \frac{1}{2}\ \mathrm{M}a^2$)

$$I_d = \frac{1}{4} M a^2$$

where M is the mass of disc and a is its radius.

Module No. 172

Example 2 of Perpendicular Axis Theorem

Problem Statement

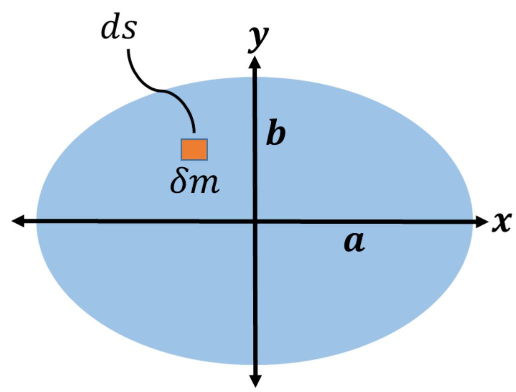
Find the M.I. of a uniform elliptical lamina with semi major axes and semi minor axes a, b respectively about respective axes (x, y, z-axis) using perpendicular axis theorem.

Proof

We consider the elliptical plate as

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad (1)$$

with semi major axes along x-axis as shown in figure.



From (1) we have

$$y=\pm\frac{b}{a}\sqrt{a^2-x^2}$$

let $y_1=\frac{b}{a}\sqrt{a^2-x^2}$

We consider a small element

of mass δm of the elliptical

plate, then we will have

$$\delta m=\rho ds=\rho dxdy$$

The moment of inertia of this element along x-axis will be equal to $I = \delta m y^2$.

Then the moment of inertia of whole plate will be

$I_{xx} = \int(\delta m)y^2 = \int_{plate} \rho ds\, y^2$

$$= \rho \int_{-a}^{a} \left(\int_{-y_1}^{y_1} y^2 dy \right) dx$$

$$= \rho \int_{-a}^{a} \frac{2y_1^3}{3} dx = \frac{2\rho}{3} \int_{-a}^{a} \frac{b^3}{a^3} (a^2 - x^2)^{3/2} dx$$

due to symmetry, we can write it as

$$= \frac{4\rho b^3}{3a^3} \int_{0}^{a} (a^2 - x^2)^{3/2}\, dx$$

By making use of polar coordinates,

substitute $x = a \sin\theta$, then $dx = a\cos\theta$ this integral becomes

$$I_{xx} = \frac{4\rho b^3}{3a^3} \int_{0}^{\pi/2} a^3 \cos^3\theta\, (a\cos\theta) d\theta$$

$$= \frac{4\rho a b^3}{3} \int_{0}^{\pi/2} \cos^4\theta\, d\theta$$

$$= \frac{1}{4} \rho a b^3 \pi$$

for elliptical plate, we have

$$\rho = \frac{M}{\pi a b}$$

Thus

$$I_{xx} = \frac{1}{4} a b^3 \pi \times \frac{M}{\pi a b}$$

$$I_{xx} = \frac{1}{4}Mb^2$$

Similarly, about y-axis

$$I_{yy} = \int (\delta m)x^2 = \int_{plate} \rho ds\, x^2$$

$$= \rho \int_{-b}^{b} \left(\int_{-x_1}^{x_1} x^2 dx \right) dy$$

$$= \rho \int_{-b}^{b} \frac{2x_1^2}{3} dy = \frac{2\rho}{3} \int_{-b}^{b} \frac{b^3}{a^3} (a^2 - y^2)^{3/2} dy$$

due to symmetry, we can write it as

$$= \frac{4\rho b^3}{3a^3} \int_0^b (a^2 - y^2)^{3/2}\, dy$$

Using polar coordinates, substitute $y = b \sin\theta$, then $dy = b\cos\theta\, d\theta$ this integral becomes

$$I_{yy} = \frac{4\rho a^3}{3b^3} \int_0^{\pi/2} b^3 \cos^3\theta\, (b\cos\theta) d\theta$$

$$= \frac{4\rho a^3 b}{3} \int_0^{\pi/2} \cos^4\theta\, d\theta$$

$$= \frac{4\rho a^3 b}{3} \frac{3}{8} \times \frac{\pi}{2}$$

$$= \frac{1}{4}\rho a^3 b\pi$$

$$= \frac{1}{4} a^3 b\pi \times \frac{M}{\pi ab}$$

$$I_{yy} = \frac{1}{4}Ma^2$$

now using perpendicular axis theorem, we obtain

$$I_{zz} = I_{xx} + I_{yy}$$

$$= \frac{1}{4}Mb^2 + \frac{1}{4}Ma^2$$

$$I_{zz} = \frac{1}{4}M(a^2 + b^2)$$

which is M.I. of elliptical lamina along z-axis

Module No. 173

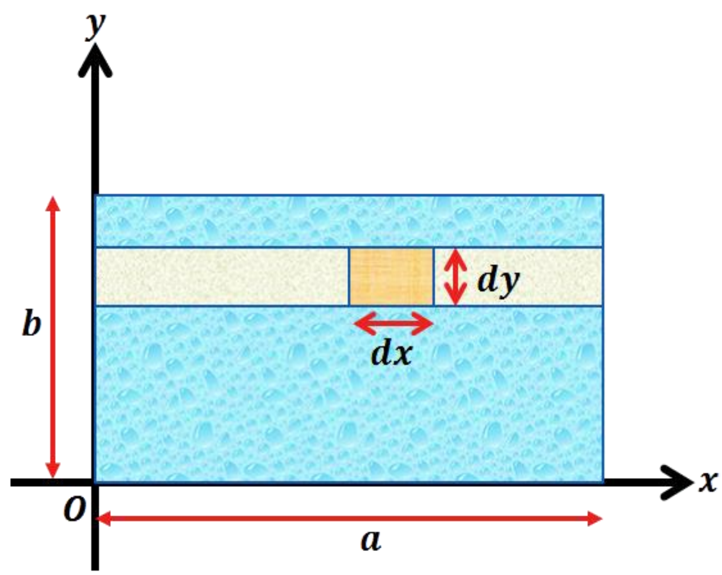
Example 3 of Perpendicular Axis Theorem

Problem

Find the moment of inertia of a rectangular plate with sides a and b about an axis perpendicular to the plate and passing through a vertex using perpendicular axis theorem.

Solution

We consider a rectangular plate (lamina) of sides of length a and b. We consider an element of length dx and dy as shown in figure.



We find the M.I about $y - axis$.

The mass of selected element will be

$$dm = \rho dxdy$$

It's moment of inertia about the $y - axis$ is

$$I_{element} = \rho dxdyx^2 = \rho x^2 dxdy.$$

where x is the perpendicular distance from the element to the $y - axis$.

Thus the total moment of inertia about y-axis is

$$I_y = \int_{x=0}^{a} \int_{y=0}^{b} \rho x^2 dx dy$$

$$= \rho b \int_{0}^{a} x^2 = \rho b \left| \frac{x^3}{3} \right|_0^a = \frac{1}{3} \rho b a^3$$

Since the density of the rectangular plate is

$$\rho = \frac{M}{ab}$$

the moment of inertia will be

$$I_y = \frac{1}{3} M a^2$$

In the similar manner, we will calculate the moment of inertia about x-axis

The total moment of inertia about x-axis is

$$I_x = \int_{x=0}^{a} \int_{y=0}^{b} \rho y^2 dx dy$$

$$= \rho a \int_{0}^{b} y^2 dy = \rho b \left| \frac{y^3}{3} \right|_0^b = \frac{1}{3} \rho a b^3$$

Using the relation of the total mass of rectangular plate

$$M = \rho ab$$

Then the moment of inertia will be

$$I_x = \frac{1}{3} M b^2$$

Thus by using perpendicular axes theorem, we obtain the moment of inertia along z-axis

$$I_z = I_x + I_y$$

$$I_z = \frac{1}{3} M b^2 + \frac{1}{3} M a^2$$

$$I_z = \frac{1}{3}M(a^2 + b^2)$$

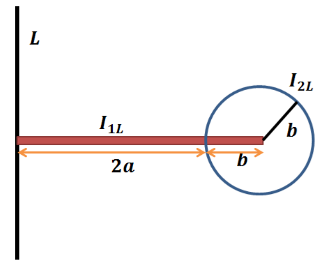
which are the required moments of inertia of rectangle along x, y, z-axis.

Module No. 174

Problem of Moment of Inertia

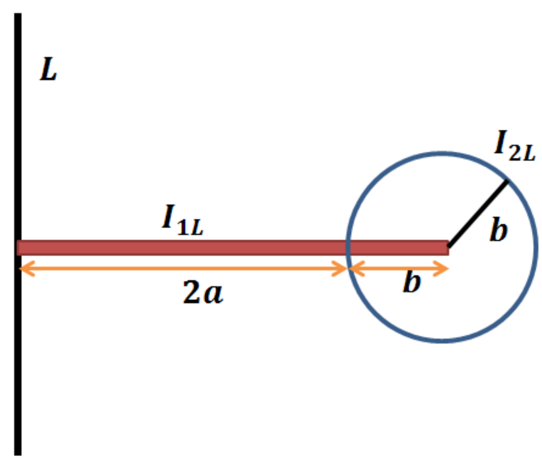
Problem

Find the moment of inertia about the line of an apparatus (as shown in figure) consisting of a sphere of mass M and radius b attached to a rod of length $2a$ & mass m.



Solution

Since from the figure it is clear that b is the radius of the sphere whose mass is M and is attached to the rod of length $2a$ whose mass is m.



Therefore

$$I_L = I_{1L} + I_{2L}$$

where I_{1L} is the M.I. of a rod of length $2a$ about L and I_{2L} is the M.I. of a sphere of radius b about the line L, then

$$I_{1L} = \int_{0}^{2a} \rho x^2 dx$$

$$= \rho \left| \frac{x^3}{3} \right|_0^{2a} = \rho \frac{8}{3} a^3$$

using relation for ρ, we get

$$I_{1L} = \frac{4}{3} m a^2 \qquad (2)$$

Now

$$I_{2L} = I' + Md^2$$

where I' is the MI of sphere about its diameter

$$I_{2L} = \frac{2}{5} Mb^2 + M(2a + b)^2 \qquad (3)$$

Substituting equation (2) & (3) in (1), we get

$$I_L = \frac{4}{3} ma^2 + \frac{2}{5} Mb^2 + M(2a + b)^2$$

which is the required M.I. of the apparatus.

Module No. 175

Existence of Principle Axes

Introduction

The axes relative to which product of inertia are zero are called the principal axes and the moment of inertia along these axes are called principal moment of inertia.

Theorem Statement

For a rigid body, there exist a set of three mutually orthogonal axes called principal axes relative to which the product of inertia are zero and $\vec{\omega}$ and L are considered along the same direction.

Proof

We assume that there exists an axis through a point O of the rigid body such that angular velocity $\vec{\omega}$ and angular momentum L of the body are parallel to this axis. Then we can write

$$L = I\vec{\omega} \qquad (1)$$

where I is a constant of proportionality.

Form equation (1), we have

$$L_1 = I\omega_1, \;\; L_2 = I\omega_2, \;\; L_3 = I\omega_3$$

Also from the general theory of angular momentum

$$L_i = \sum_j \, I_{ij}\, \omega_j \qquad (2)$$

From equation (1) and (2), we have

$$L_1 = I_{11}\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 = I\omega_1$$

$$L_2 = I_{21}\omega_1 + I_{22}\omega_2 + I_{23}\omega_3 = I\omega_2$$

$$L_3 = I_{31}\omega_1 + I_{32}\omega_2 + I_{33}\omega_3 = I\omega_3$$

which can also be written as

$$(I_{11} - I)\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 = 0$$

$$I_{21}\omega_1 + (I_{22} - I)\omega_2 + I_{23}\omega_3 = 0$$

$$I_{31}\omega_1 + I_{32}\omega_2 + (I_{33} - I)\omega_3 = 0 \qquad (3)$$

Equation (3) is a system of three homogeneous linear algebraic equation in the unknown $\omega_1, \omega_2, \omega_3$.

This system will have a non-trivial solution, i.e. $(\vec{\omega} \neq 0)$, if the determinant of the matrix of coefficients is zero. i.e.

$$\begin{vmatrix} I_{11} - I & I_{12} & I_{13} \\ I_{21} & I_{22} - I & I_{23} \\ I_{31} & I_{32} & I_{33} - I \end{vmatrix} = 0 \qquad (4)$$

This is a cubic equation in I and will in general have three roots. Equation (4) is called characteristic equation of the matrix (I_{ij}).

The roots of equation (4) are called eigen values of the inertia matrix (I_{ij}).

The problem of finding principal moment of inertia and directions of inertia has been reduced to that of finding the eigenvalues and eigenvectors of a symmetric 3×3 matrix.

The following results from the eigenvalue theory of matrices will deduce further results about the principal moments and directions (axes) of inertia.

Related Theorems

- **Theorem 1**

A 3×3 symmetrical matrix has three real eigenvalues, which may be distinct or repeated.

- **Theorem 2**

The eigenvectors of a symmetrical matrix corresponding to distinct eigenvalues are orthogonal.

- **Theorem 3**

It is always possible to find three mutually orthogonal eigenvectors for a 3×3 symmetric matrix, whether the eigenvalues are distinct or repeated.

Results

In the light of above theorems, we deduce the following results about the inertia matrix I_{mat} at point O of a rigid body.

i. The principal of moment of inertia are always real numbers. This is obviously physical because the moment of inertia is defined as the quantity $\sum_i m_i d_i^2$ where m_i and d_i are both real.

ii. When all three principal moments I_1, I_2, I_3 are distinct, then by the theorem 2, three mutually orthogonal principal axes can be found.

iii. When $I_1 = I_2$ but $I_1 \neq I_3$, then by the theorem 3, we can still determine three mutually orthogonal principal axes, because the inertia matrix is symmetric.

Module No. 176

Determination of Principal Axes of Other Two When One is known

In many instances a body possesses sufficient symmetry so that at least one principal axis can be found by inspection, i.e;the axis can be chosen so as to make two of the three products of inertia vanish.

We consider a plane rigid body i.e. a 2D body; for example a plate of uniform thickness. Such a system can be regarded as a coplanar distribution of mass.

Since there are three mutually orthogonal principal axes, one of them must be perpendicular to the plane of the body.

The other two axes will lie in the plane of lamina.

We choose X and Y axes in the plane of lamina, and the Z axis perpendicular to its plane, as shown in figure.

$I_{xz} = I_{yz} = 0$ whereas $I_{xy} \neq 0$ (1)

Since we have obtained the following equations for principal axes.

$$(I_{11} - I)\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 = 0$$

$$I_{21}\omega_1 + (I_{22} - I)\omega_2 + I_{23}\omega_3 = 0$$

$$I_{31}\omega_1 + I_{32}\omega_2 + (I_{33} - I)\omega_3 = 0 \qquad (2)$$

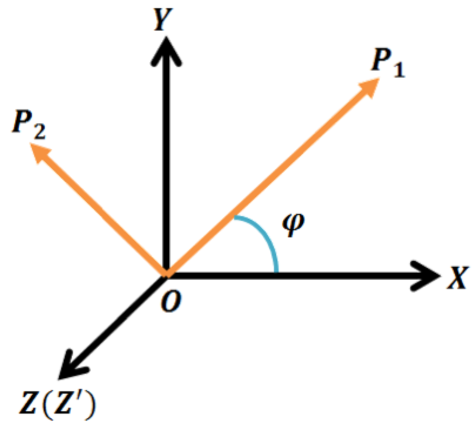
By making use of (1), and using the first two equations of (2), we obtain

$$(I_{11} - I)\omega_1 + I_{12}\omega_2 = 0$$

$$I_{21}\omega_1 + (I_{22} - I)\omega_2 = 0$$

Since the principal axes perpendicular to the plane of lamina is supposed to be known, and we are interested in determining the principal axes in XY-plane, therefore we didn't consider third eq. of (2).

Let $OP_1,\ OP_2$ denote the principal axes in the XY-plane and let φ denote the angle between the OP_1 and the $X - axis$.



We define

$$\tan\varphi = \omega_2/\omega_1$$

which can also be written as

$$\frac{\omega_1}{\cos\varphi} = \frac{\omega_2}{\sin\varphi} = k, \qquad (3)$$

where k is any arbitrary constant.

Substituting for $\omega_1,\ \omega_2$ from equation (3) in (2) we have, after simplification

$$(I_{11} - I)\cos\varphi + I_{12}\sin\varphi = 0$$

$$I_{21}\cos\varphi + (I_{22} - I)\sin\varphi = 0$$

These equations can be put in the form

$$I_{11} - I = -I_{12}\frac{\sin\varphi}{\cos\varphi},\ \ I_{22} - I = -I_{12}\frac{\cos\varphi}{\sin\varphi}$$

Module No. 177

Determination of Principal Axes by Diagonalizing the Inertia Matrix

Introduction

Suppose a rigid body has no axis of symmetry. Even so, the tensor that represents the moment of inertia of such a body, is characterized by a real, symmetric 3×3 matrix that can be diagonalized. The resulting diagonal elements are the values of the principal moments of inertia of the rigid body.

The axes of the coordinate system, in which this matrix is diagonal, are the principal axes of the body, because all products of inertia have vanished.

Thus, finding the principal axes and corresponding moments of inertia of any rigid body, symmetric or not, is virtually the same as to diagonalzing its moment of inertia matrix.

Explanation

There are a number of ways to diagonalize a real, symmetric matrix. We present here a way that is quite standard.

First, suppose that we have found the coordinate system (principal axes) in which all products of inertia vanish and the resulting moment of inertia tensor is now represented by a diagonal matrix whose diagonal elements are the principal moments of inertia.

Let e_i be the unit vectors that represent this coordinate system, that is, they point along the direction along the three principal axes of the rigid body.

If the moment of inertia tensor is "dotted" with one of these unit vectors, the result is equivalent to a simple multiplication of the unit vector by a scalar quantity, i.e.

$$Ie_i = \lambda e_i \tag{1}$$

The quantities λ_i are just the principal M.I about their respective principal axes.The problem of finding the principal axes is one of finding those vectors e_i that satisfy the condition

$$(I - \lambda I)e_i = 0 \tag{2}$$

In general this condition is not satisfied for any arbitrary set of orthonormal unit vectors e_i. It is satisfied only by a set of unit vectors aligned with the principal axes of the rigid body.

Any arbitrary xyz coordinate system can always be rotated such that the coordinate axes line up with the principal axes. The unit vectors specifying these coordinate axes then satisfy the condition in eq (2). This condition is equivalent to vanishing of the following determinant

$$|I - \lambda I| = 0 \qquad (3)$$

Explicitly, this equation reads

$$\begin{vmatrix} I_{11} - I & I_{12} & I_{13} \\ I_{21} & I_{22} - I & I_{23} \\ I_{31} & I_{32} & I_{33} - I \end{vmatrix} = 0$$

It is a cubic in λ, namely,

$$-\lambda^3 + A\lambda^2 + B\lambda + C = 0 \qquad (4)$$

in which A, B, and C are functions of the I's. The three roots λ_1, λ_2 and λ_3 are the three principal moments of inertia.

We now have the principal moments of inertia, but the task of specifying the components of the unit vectors representing the principal axes in terms of our initial coordinate system remains to be solved.

Here we can make use of the fact that when the rigid body rotates about one of its principal axes; the angular momentum vector is in the same direction as the angular velocity vector.

Let the angles of one of the principal axes relative to the initial xyz coordinate system be α, β and γ and let the body rotate about this axis. Therefore, a unit vector pointing in the direction of this principal axis has components $(\cos\alpha, \cos\beta, \cos\gamma)$.

Using eq (1),

$$Ie_1 = \lambda_1 e_1$$

where λ_1, the first principal moment of the three $(\lambda_1, \lambda_2, \lambda_3)$, is obtained by solving eq (4).

In matrix form

$$\begin{bmatrix} I_{11} - \lambda_1 & I_{12} & I_{13} \\ I_{21} & I_{22} - \lambda_1 & I_{23} \\ I_{31} & I_{32} & I_{33} - \lambda_1 \end{bmatrix} \begin{bmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{bmatrix} = 0$$

- The direction cosines may be found by solving the above equations.
- The solutions are not independent. They are subject to the constraint

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

- In other words the resultant vector e_1 specified by these components is a unit vector.

Module No. 178

Relation of Fixed and Rotating Frames of Reference

In order to derive the relationship between fixed and rotating frames of reference, we will study the following theorem[1,2].

Rotating Axes Theorem

- **Theorem Statement**

If a time dependent vector function $\vec{A}$ is represented by $\vec{A}_f$ and $\vec{A}_r$ in fixed and rotating coordinate system, then

$$\left(\frac{d\vec{A}}{dt}\right)_f = \left(\frac{d\vec{A}}{dt}\right)_r + \omega \times \vec{A}_r$$

where it is understood that the origins of the two systems coincide at $t = 0$

- **Proof**

We denote the fixed and rotating coordinate systems by $OX_0Y_0Z_0$ and $OXYZ$ and denote the associated unit vectors by $\{i_0, j_0, k_0\}$ and $\{i, j, k\}$.

Consider a vector A which is changing with time. To an observer fixed relative to $OXYZ$ system, the time rate of change of $A = A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k}$ will be

$$\frac{d}{dt}A_r = \frac{d}{dt}A_1\hat{\imath} + \frac{d}{dt}A_2\hat{\jmath} + \frac{d}{dt}A_3\hat{k}$$

where $\frac{dA_r}{dt}$ denotes the time derivative of A relative to the rotating frame of reference.

However, the time rate of change of A relative to the fixed system $OX_0Y_0Z_0$ symbolized by the $\frac{dA_f}{dt}$ needs to be found.

To the fixed observer the unit vectors $i,\ j,\ k$ actually change with time.

Thus

$$\frac{d}{dt}A_f = \frac{d}{dt}(A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k})$$

$$= \frac{dA_1}{dt}\hat{\imath} + \frac{dA_2}{dt}\hat{\jmath} + \frac{dA_3}{dt}\hat{k} + A_1\frac{d\hat{\imath}}{dt} + A_2\frac{d\hat{\jmath}}{dt} + A_3\frac{d\hat{k}}{dt}$$

$$\frac{dA_f}{dt} = \frac{dA_r}{dt} + A_1\frac{d\hat{\imath}}{dt} + A_2\frac{d\hat{\jmath}}{dt} + A_3\frac{d\hat{k}}{dt} \qquad (1)$$

Since $\hat{\imath}$ is a unit vector, $d\hat{\imath}/dt$ is perpendicular to $\hat{\imath}$. Then $\frac{d\hat{\imath}}{dt}$ must lie in the plane of $\hat{\jmath}$ and $\hat{k}$. Therefore

$$\frac{d\hat{\imath}}{dt} = \alpha_1\hat{\jmath} + \alpha_2\hat{k} \qquad (2)$$

Similarly,

$$\frac{d\hat{\jmath}}{dt} = \alpha_3\hat{k} + \alpha_4\hat{\imath} \qquad (3)$$

$$\frac{d\hat{k}}{dt} = \alpha_5\hat{\imath} + \alpha_6\hat{\jmath} \qquad (4)$$

Form $\hat{\imath}.\hat{\jmath} = 0$, differentiation yields

$$\hat{\imath}.\frac{d\hat{\jmath}}{dt} + \hat{\jmath}.\frac{d\hat{\imath}}{dt} = 0 \Longrightarrow \hat{\imath}.\frac{d\hat{\jmath}}{dt} = -\hat{\jmath}.\frac{d\hat{\imath}}{dt}$$

But from (2), we have

$$\hat{\jmath}.\frac{d\hat{\imath}}{dt} = \alpha_1 \text{ and } \hat{\imath}.\frac{d\hat{\jmath}}{dt} = \alpha_4$$

$$\Longrightarrow \alpha_4 = -\alpha_1$$

Similarly form $\hat{\imath}.\hat{k} = 0$ we obtain

$$\hat{\imath}.\frac{d\hat{k}}{dt} + \hat{k}.\frac{d\hat{\imath}}{dt} = 0 \Longrightarrow \hat{\imath}.\frac{d\hat{k}}{dt} = -\hat{k}.\frac{d\hat{\imath}}{dt}$$

From (3), we have

$$\hat{k}.\frac{d\hat{\imath}}{dt} = \alpha_2 \text{ and } \hat{k}.\frac{d\hat{\jmath}}{dt} = \alpha_5$$

$$\Longrightarrow \alpha_5 = -\alpha_2$$

and from $\hat{\jmath}.\hat{k} = 0$ we obtain

$$\hat{\jmath}.\frac{d\hat{k}}{dt} + \hat{k}.\frac{d\hat{\jmath}}{dt} = 0 \Longrightarrow \hat{\jmath}.\frac{d\hat{k}}{dt} = -\hat{k}.\frac{d\hat{\jmath}}{dt}$$

From (4), we have

$\hat{k}.\frac{d\hat{\jmath}}{dt} = \alpha_2$ and $\hat{\jmath}.\frac{d\hat{k}}{dt} = \alpha_5$

$$\Longrightarrow \alpha_6 = -\alpha_3$$

Then

$$\frac{d\hat{\imath}}{dt} = \alpha_1\hat{\jmath} + \alpha_2\hat{k}$$

$$\frac{d\hat{\jmath}}{dt} = \alpha_3\hat{k} - \alpha_1\hat{\imath}$$

$$\frac{d\hat{k}}{dt} = -\alpha_2\hat{\imath} - \alpha_3\hat{\jmath}$$

follows that

$$A_1\frac{d\hat{\imath}}{dt} + A_2\frac{d\hat{\jmath}}{dt} + A_3\frac{d\hat{k}}{dt} = (-\alpha_1A_2 - \alpha_2A_3)\hat{\imath} + (-\alpha_1A_1 - \alpha_3A_3)\hat{\jmath} + (-\alpha_2A_1 - \alpha_3A_2)\hat{k}$$

where

$$\vec{\omega} = \omega_1\hat{\imath} + \omega_2\hat{\jmath} + \omega_3\hat{k}$$

The vector quantity $\vec{\omega}$ is the angular velocity of the moving system relative to the fixed system.

Thus from (1) and (5), we obtain

$$\left(\frac{d\vec{A}}{dt}\right)_f = \left(\frac{d\vec{A}}{dt}\right)_r + \omega \times \vec{A}$$

Module No. 179

Equation of Motion in Rotating Frame of Reference

There are two cases to be discussed in this article. Frist is when the origins of fixed and rotating coordinate system coincide and other is when the origins of two system are distant.

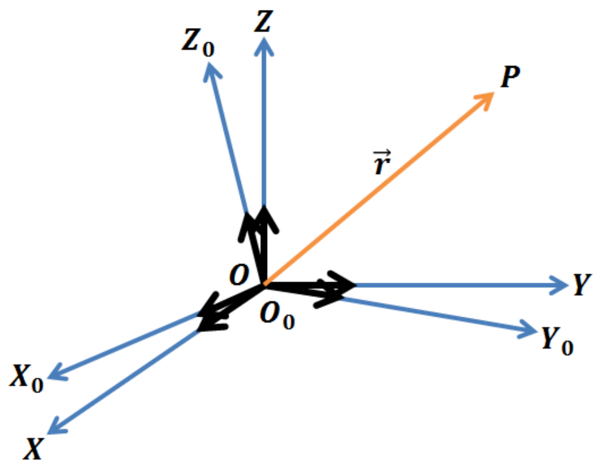
Case I

In this case we consider the origins of the fixed and rotating coordinate system coincide. This case was earlier discussed in detail, where it was found that:

To an observer fixed relative to $OXYZ$ system, the time rate of change of $r = r_1\hat{\imath} + r_2\hat{\jmath} + r_3\hat{k}$ will be

$$\frac{d}{dt}r_r = \frac{d}{dt}r_1\hat{\imath} + \frac{d}{dt}r_2\hat{\jmath} + \frac{d}{dt}r_3\hat{k} \qquad (1)$$

where $\frac{dr_r}{dt}$ denotes the time derivative of r relative to the rotating frame of reference.



However, the time rate of change of r relative to fixed system $OX_0Y_0Z_0$ symbolized by $\frac{dr_f}{dt}$ will be

$$\frac{dr_f}{dt} = \left(\frac{dr}{dt}\right)_f = \left(\frac{dr}{dt}\right)_r + \omega \times r \qquad (2)$$

or in operator form, we can write

$$\left(\frac{d}{dt}\right)_f = \left(\frac{d}{dt}\right)_r + \omega \times$$

Differentiating both sides of (2) w.r.t the fixed coordinate system, we have

$$\left(\frac{d}{dt}\right)_f v_f = \left(\frac{d}{dt}\right)_f (v_r + \omega \times r)$$

by applying operator, we have

$$\left(\frac{d}{dt}\right)_f v_f = \left(\frac{d}{dt} + \omega \times\right)_r (v_r + \omega \times r)$$

$$= a_r + 2\omega \times \frac{dr}{dt} + \dot{\omega} \times r + \omega \times (\omega \times r)$$

or

$a_f = a_r + 2\omega \times v_r + \dot{\omega} \times r + \omega \times (\omega \times r)$
(3)

where

$$a_f = \frac{dv_f}{dt}, \qquad a_r = \frac{dv_r}{dt}$$

are the acceleration in the fixed and rotating coordinate systems. The relation (3) is referred as Coriolis theorem. The term $2\omega \times v_r$ is called Coriolis acceleration, whereas the term $\omega \times (\omega \times r)$ is called centripetal acceleration.

The equations of motion in fixed and moving/ rotating coordinate system are

$$F = ma_f, \qquad F' = ma_r$$

where F and F' are the total forces in the fixed and the rotating coordinate systems.

On making substitution for a_f from (3) in equation $F = ma_f$, we obtain

These forces are called fictitious or apparent or inertial forces. They do not have physical forces and do not arise from interactions of the particles.

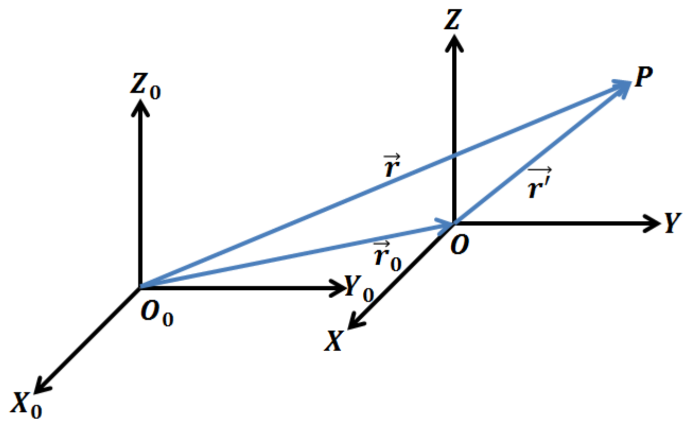
Case II

In this case we consider the origins of fixed and rotating coordinate systems are not coincident. We have

$$(r)_f = R = (r)_r + r_0$$

$$= x\hat{\imath} + y\hat{\jmath} + z\hat{k} + r_0$$

where r_0 is the position vector of the origin of the rotating system w.r.t the origin of the fixed system.



Therefore

$$\left(\frac{dr}{dt}\right)_f = \frac{dr_0}{dt} + \frac{d}{dt}(x\hat{\imath} + y\hat{\jmath} + z\hat{k})$$

$$v_f = v_0 + (\dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{k}) + x\frac{d\hat{\imath}}{dt} + y\frac{d\hat{\jmath}}{dt} + z\frac{d\hat{k}}{dt}$$

$$v_f = v_0 + v_r + \omega \times r$$

where $r \equiv (r)_{r.}$

Similarly the acceleration in the two systems will be related by

$$a_f = a_0 + a_r + 2\omega \times v_r + \dot{\omega} \times r + \omega \times (\omega \times r)$$

Module No. 180

Example 1 of Equation of Motion in Rotating Frame of Reference

Problem Statement

The angular velocity of a rotating coordinate system $OXYZ$ relative to a fixed coordinate system $OX_0Y_0Z_0$ is

$$\vec{\omega} = 2t\hat{\imath} - t^2\hat{\jmath} + (2t+4)\hat{k}$$

where t is the time and $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ unit vectors associated with $OXYZ$. The position vector of a particle at time t in the body system ($OXYZ$ system) is given by

$$r = (t^2+1)\hat{\imath} - 6t\hat{\jmath} + 4t^3\hat{k}$$

To Find

i. The apparent and true velocities at time $t = 1$.

ii. The apparent and true acceleration at time $t = 1$

Solution

The apparent velocity is given by

$$v_r = \left(\frac{dr}{dt}\right)_r = \frac{d}{dt}\left[(t^2+1)\hat{\imath} - 6t\hat{\jmath} + 4t^3\hat{k}\right]$$

$$= \hat{\imath}\frac{d}{dt}(t^2+1) - \hat{\jmath}\frac{d}{dt}6t + \hat{k}\frac{d}{dt}4t^3$$

$$= 2t\hat{\imath} - 6\hat{\jmath} + 12t^2\hat{k}$$

Therefore the apparent velocity at time $t = 1$ is given by

$$v_r(t=1) = 2\hat{\imath} - 6\hat{\jmath} + 12\hat{k}$$

The true velocity at any time t is given by

$$v_f = v_r + \vec{\omega} \times r$$

$$= 2\hat{\imath} - 6\hat{\jmath} + 12\hat{k} + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2t & -t^2 & 2t+4 \\ t^2+1 & -6t & 4t^3 \end{vmatrix}$$

Therefore

$$v_f(t = 1) = 2\hat{\imath} - 6\hat{\jmath} + 12\hat{k} + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 6 \\ 2 & -6 & 4 \end{vmatrix}$$

$$= 34\hat{\imath} - 2\hat{\jmath} + 2\hat{k}$$

ii. For apparent acceleration

$$a_r = \frac{dv_r}{dt} = \frac{d}{dt}\left(2t\hat{\imath} - 6\hat{\jmath} + 12t^2\hat{k}\right)$$

$$= 2\hat{\imath} + 24t\hat{k}$$

Therefore

$$a_r(t = 1) = 2\hat{\imath} + 24\hat{k}$$

For true acceleration we have

$a_f = a_r + 2\omega \times v_r + \dot{\omega} \times r + \omega \times (\omega \times r)$
(1)

now

$$\omega(t = 1) = 2\hat{\imath} - \hat{\jmath} + 6\hat{k}$$

and

$$\dot{\omega} = 2\hat{\imath} - 2t\hat{\jmath} + 2\hat{k}$$

$$\dot{\omega}(t = 1) = 2\hat{\imath} - 2\hat{\jmath} + 2\hat{k}$$

Since $r = (t^2 + 1)\hat{\imath} - 6t\hat{\jmath} + 4t^3\hat{k}$ and

$v_r = 2t\hat{\imath} - 6\hat{\jmath} + 12t^2\hat{k}$, therefore

$$\dot{\omega} \times r = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -2 & 2 \\ 2 & -6 & 4 \end{vmatrix} = 4\hat{\imath} - 4\hat{\jmath} - 8\hat{k}$$

and

$$2\omega \times v_r = 2\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 6 \\ 2 & -6 & 12 \end{vmatrix}$$

$$= 2(24\hat{\imath} - 12\hat{\jmath} - 10\hat{k})$$

Also since

$$\omega \times r = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 6 \\ 2 & -6 & 4 \end{vmatrix} = 32\hat{\imath} + 4\hat{\jmath} - 10\hat{k}$$

Therefore

$$\omega \times (\omega \times r) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 6 \\ 32 & 4 & -10 \end{vmatrix}$$

$$= -14\hat{\imath} + 212\hat{\jmath} + 40\hat{k}$$

Hence on making substitution, we have

$$a_f = \left(2\hat{\imath} + 24t\hat{k}\right) + \left(4\hat{\imath} - 4\hat{\jmath} - 8\hat{k}\right) + 2\left(24\hat{\imath} - 12\hat{\jmath} - 10\hat{k}\right) + (-14\hat{\imath} + 212\hat{\jmath} + 40\hat{k})$$

$$= 40\hat{\imath} + 184\hat{\jmath} + 36\hat{k}$$

which is required true acceleration.

Module No. 181

Example 2 of Equation of Motion in Rotating Frame of Reference

Problem Statement

A coordinate system OXYZ rotates with angular velocity $\vec{\omega} = 2\hat{\imath} - 3\hat{\jmath} + 5\hat{k}$ relative to the fixed coordinate system O$X_0Y_0Z_0$ both systems having the same origin. If $\vec{A} = \sin t\,\hat{\imath} - \cos t\,\hat{\jmath} + e^{-t}\hat{k}$ where $\hat{\imath}, \hat{\jmath}, \hat{k}$ refer to the rotating coordinate system OXYZ then find $\frac{dA}{dt}$ and $\frac{d^2A}{dt^2}$ w.r.to

i. Fixed system

ii. The rotating system

Solution

i. To find $\frac{dA}{dt}$ and $\frac{d^2A}{dt^2}$ in the fixed system, we have to apply the following formulas

$$\left(\frac{d\vec{A}}{dt}\right)_f = \left(\frac{d\vec{A}}{dt}\right)_r + \vec{\omega} \times \vec{A} \tag{1}$$

$$\left(\frac{d^2\vec{A}}{dt^2}\right)_f = \left(\frac{d^2\vec{A}}{dt^2}\right)_r + 2\vec{\omega} \times \frac{d\vec{A}}{dt} + \frac{d\vec{\omega}}{dt} \times \vec{A} + \vec{\omega} \times (\vec{\omega} \times \vec{A}) \tag{2}$$

$$\left(\frac{d\vec{A}}{dt}\right)_r = \left[\frac{d\vec{A}}{dt}\right]_r = \frac{d\vec{A}_r}{dt}$$

$$= \frac{d}{dt}(\sin t\,\hat{\imath} - \cos t\,\hat{\jmath} + e^{-t}\hat{k})$$

$$= \cos t\,\hat{\imath} + \sin t\,\hat{\jmath} - e^{-t}\hat{k}$$

$$\vec{\omega} \times \vec{A} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -3 & 5 \\ \sin t & -\cos t & e^{-t} \end{vmatrix}$$

$$\vec{\omega} \times \vec{A} = (-3e^{-t} + 5\cos t)\hat{\imath} - (2e^{-t} - 5\sin t)\hat{\jmath} + (-2\cos t + 3\sin t)\hat{k}$$

$$\left(\frac{d\vec{A}}{dt}\right)_f = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath} - e^{-t}\hat{k} + (-3e^{-t} + 5\cos t)\hat{\imath} - (2e^{-t} - 5\sin t)\hat{\jmath}$$
$$+ (-2\cos t + 3\sin t)\hat{k}$$

$$= (\cos t + 5\cos t - 3e^{-t})\hat{\imath} + (5\sin t + \sin t - 2e^{-t})\hat{\jmath} + (-e^{-t} - 2\cos t + 3\sin t)\hat{k}$$

$$= (6\cos t - 3e^{-t})\hat{\imath} + (6\sin t - 2e^{-t})\hat{\jmath} + (-e^{-t} - 2\cos t + 3\sin t)\hat{k}$$

Now for solving equation (2), we proceed as follows

$$\left(\frac{d^2\vec{A}}{dt^2}\right)_r = \frac{d}{dt}\left(\frac{d\vec{A}}{dt}\right)_r = \frac{d}{dt}(\cos t\,\hat{\imath} + \sin t\,\hat{\jmath} - e^{-t}\hat{k})$$

$$= (-\sin t\,\hat{\imath} + \cos t\,\hat{\jmath} + e^{-t}\hat{k})$$

$$2\vec{\omega} \times \left(\frac{d\vec{A}}{dt}\right)_r = 2(2\hat{\imath} - 3\hat{\jmath} + 5\hat{k}) \times (\cos t\,\hat{\imath} + \sin t\,\hat{\jmath} - e^{-t}\hat{k})$$

$$= (4\hat{\imath} - 6\hat{\jmath} + 10\hat{k}) \times (\cos t\,\hat{\imath} + \sin t\,\hat{\jmath} - e^{-t}\hat{k})$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4 & -6 & 10 \\ \cos t & \sin t & -e^{-t} \end{vmatrix}$$

$$2\vec{\omega} \times \left(\frac{d\vec{A}}{dt}\right)_r$$

$$= (6e^{-t} - 10\sin t)\hat{\imath} + (4e^{-t} + 10\cos t)\hat{\jmath} + (4\sin t + 6\cos t)\hat{k}$$

$$\vec{\omega} = 2\hat{\imath} - 3\hat{\jmath} + 5\hat{k} \Longrightarrow \dot{\vec{\omega}} = \frac{d\vec{\omega}}{dt} = 0$$

So,

$$\frac{d\vec{\omega}}{dt} \times \vec{A} = 0 \times \left(\sin t\,\hat{\imath} - \cos t\,\hat{\jmath} + e^{-t}\hat{k}\right) = 0$$

$$\vec{\omega} \times \left(\vec{\omega} \times \vec{A}\right) = (2\hat{\imath} - 3\hat{\jmath} + 5\hat{k}) \times \left[(2\hat{\imath} - 3\hat{\jmath} + 5\hat{k}) \times (\sin t\,\hat{\imath} - \cos t\,\hat{\jmath} + e^{-t}\hat{k})\right]$$

$$= (2\hat{\imath} - 3\hat{\jmath} + 5\hat{k}) \times \left[(-3e^{-t} + 5\cos t)\hat{\imath} - (2e^{-t} - 5\sin t)\hat{\jmath} + (-2\cos t + 3\sin t)\hat{k}\right]$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -3 & 5 \\ -3e^{-t} + 5\cos t & -2e^{-t} + 5\sin t & -2\cos t + 3\sin t \end{vmatrix}$$

$$= (6\cos t - 9\sin t - 25\sin t + 10e^{-t})\hat{\imath} + (4\cos t - 6\sin t - 15e^{-t} + 25\cos t)\hat{\jmath} + (4e^{-t} - 10\sin t + 9e^{-t} - 15\cos t)\hat{k}$$

$$= (6\cos t - 34\sin t + 10e^{-t})\hat{\imath} + (29\cos t - 6\sin t - 15e^{-t})\hat{\jmath} + (-4e^{-t} + 10\sin t + 9e^{-t} - 15\cos t)\hat{k}$$

$$= \left(-\sin t\,\hat{\imath} + \cos t\,\hat{\jmath} + e^{-t}\hat{k}\right) + (6e^{-t} - 10\sin t)\hat{\imath} + (4e^{-t} + 10\cos t)\hat{\jmath} + (4\sin t + 6\cos t)\hat{k} + 0 + (6\cos t - 34\sin t + 10e^{-t})\hat{\imath} + (29\cos t - 6\sin t - 15e^{-t})\hat{\jmath} + (10\sin t - 13e^{-t} + 15\cos t)\hat{k}$$

$$\left(\frac{d^2\vec{A}}{dt^2}\right)_f = (6\cos t - 45\sin t + 16e^{-t})\hat{\imath} + (40\cos t - 6\sin t - 11e^{-t})\hat{\jmath} + (14\sin t - 12e^{-t} + 21\cos t)\hat{k}$$

ii. The rotating system

$$\left(\frac{d\vec{A}}{dt}\right)_r = \left[\frac{d\vec{A}}{dt}\right]_r = \frac{d\vec{A}_r}{dt}$$

$$= \frac{d}{dt}(\sin t\,\hat{\imath} - \cos t\,\hat{\jmath} + e^{-t}\hat{k})$$

$$= \cos t\,\hat{\imath} + \sin t\,\hat{\jmath} - e^{-t}\hat{k}$$

$$\left(\frac{d^2\vec{A}}{dt^2}\right)_r = \frac{d}{dt}\left(\frac{d\vec{A}}{dt}\right)_r$$

$$= \frac{d}{dt}(\cos t\,\hat{\imath} + \sin t\,\hat{\jmath} - e^{-t}\hat{k})$$

$$= (-\sin t\,\hat{\imath} + \cos t\,\hat{\jmath} + e^{-t}\hat{k})$$

Module No. 182

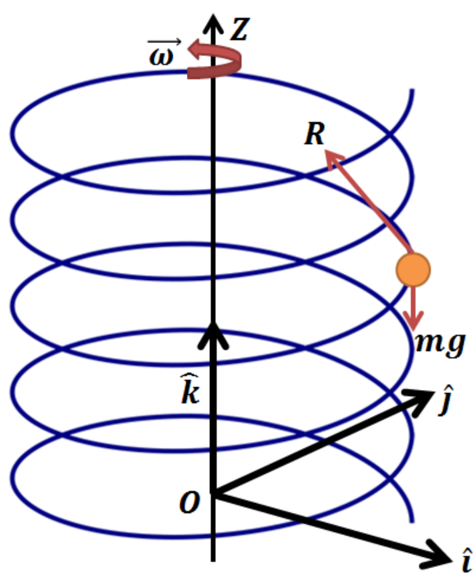
Example 3 of Equation of Motion in Rotating Frame of Reference

Problem Statement

A bead slides on a smooth helix whose central axis is vertical. The helix is forced to rotate about its central axis with constant angular speed ω. Find the equation of motion of the bead relative to the helix.

Solution

We choose a coordinate $OXYZ$ fixed in the helix such that Z-axis is coincident with the axis of the helix.



Then the parametric eqs. of the helix are given by

$$x = a\cos\theta\,, y = a\sin\theta,$$

$$z = b\theta$$

In vector form these can be written as

$$\vec{r} = a\cos\theta\,\hat{\imath} + a\sin\theta\,\hat{\jmath} + b\theta\hat{k}$$

where $\hat{\imath}, \hat{\jmath}, \hat{k}$ point along the coordinate axis. Here $\hat{k}$ is constant but $\hat{\imath}, \hat{\jmath}$ are variable.

The equation of motion in the rotating coordinate system is given by

$$ma_r = F - 2m\omega \times v_r - m\dot{\omega} \times r - m\omega \times (\omega \times r) \quad (1)$$

In this problem, since ω is constant,

$$\omega = \omega\hat{k}\,, \qquad \dot{\omega} = 0 \quad (2)$$

Also

$$v_r = \dot{r} = (-a\sin\theta\,\hat{\imath} + a\cos\theta\,\hat{\jmath} + b\hat{k})\dot{\theta} \quad (3)$$

and

$$a_r = \ddot{r} = \frac{d}{dt}\left[\left(-a\sin\theta\,\hat{\imath} + a\cos\theta\,\hat{\jmath} + b\hat{k}\right)\dot{\theta}\right]_r$$

$$= \left(-a\sin\theta\,\hat{\imath} + a\cos\theta\,\hat{\jmath} + b\hat{k}\right)\ddot{\theta} + (-a\cos\theta\,\hat{\imath} - a\sin\theta\,\hat{\jmath})\dot{\theta}^2 \quad (4)$$

If F denotes the external force in the fixed (space) coordinate system, then

$$F = -mg\hat{k} + R \quad (5)$$

where R is the reaction of the helix on the bead.

Before making substitution from equation (2-5) into equation (1), we obtain simplified expression for the second, and the fourth terms of (1), (the third term being zero).

$$\omega \times v_r = \omega\hat{k} \times (-a\sin\theta\,\hat{\imath} + a\cos\theta\,\hat{\jmath} + b\hat{k})\dot{\theta}$$

$$= \omega a(-\sin\theta\,\hat{\jmath} - \cos\theta\,\hat{\imath})\dot{\theta} \quad (6)$$

and

$$\omega \times r = \omega\hat{k} \times \left(a\cos\theta\,\hat{\imath} + a\sin\theta\,\hat{\jmath} + b\theta\hat{k}\right)$$

$$= \omega a(\cos\theta\,\hat{\jmath} - \sin\theta\,\hat{\imath})$$

$$\omega \times (\omega \times r) = \omega^2 a\hat{k} \times (\cos\theta\,\hat{\jmath} - \sin\theta\,\hat{\imath})$$

$$= \omega^2 a(-\cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath}) \quad (7)$$

Therefore on making substitutions from (4), (5), (6) and (7) into (1), we obtain

$$m\left(-a\sin\theta\,\hat{\imath} + a\cos\theta\,\hat{\jmath} + b\hat{k}\right)\ddot{\theta} - ma(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) = -mg\hat{k} + R + 2m\omega a(\sin\theta\,\hat{\jmath} + \cos\theta\,\hat{\imath}) + ma\omega^2(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})$$

$$m\left(-a\sin\theta\,\hat{\imath}+a\cos\theta\,\hat{\jmath}+b\hat{k}\right)\ddot{\theta}-ma(\cos\theta\,\hat{\imath}+\sin\theta\,\hat{\jmath})=-mg\hat{k}+R+(2m\omega a+ma\omega^2)\,(\sin\theta\,\hat{\jmath}+\cos\theta\,\hat{\imath}) \qquad (8)$$

Now if we write

$$v_r=\left(-a\sin\theta\,\hat{\imath}+a\cos\theta\,\hat{\jmath}+b\hat{k}\right)\dot{\theta}=u\dot{\theta}$$

Then it is clear that the vector u is tangent to the helix. Since R is normal to the helix,

$$R.\,v_r=R.u\dot{\theta}=0$$

Rewriting (8) in terms of u (after eliminating the common factor m), we have

$$u\ddot{\theta}-a(\cos\theta\,\hat{\imath}+\sin\theta\,\hat{\jmath})\dot{\theta}^2=-gk+R+(2\omega a+a\omega^2)(\cos\theta\,\hat{\imath}+\sin\theta\,\hat{\jmath}) \qquad (9)$$

Taking dot product of both sides of (9) with u, we have

$$u.u\ddot{\theta}-au.(\cos\theta\,\hat{\imath}+\sin\theta\,\hat{\jmath})\dot{\theta}^2=-gk.u+R.u+(2\omega a+a\omega^2)(\cos\theta\,\hat{\imath}+\sin\theta\,\hat{\jmath}).u \qquad (10)$$

Now

$$u.u=a^2\sin^2\theta+a^2\cos^2\theta+b^2=a^2+b^2$$

$$\hat{k}.u=\hat{k}.\left(-a\sin\theta\,\hat{\imath}+a\cos\theta\,\hat{\jmath}+b\hat{k}\right)=b$$

$$R.u=0$$

$$(\sin\theta\,\hat{\jmath}+\cos\theta\,\hat{\imath}).u=(\sin\theta\,\hat{\jmath}+\cos\theta\,\hat{\imath}).\left(-a\sin\theta\,\hat{\imath}+a\cos\theta\,\hat{\jmath}+b\hat{k}\right)$$

$$=a\cos\theta\sin\theta-a\cos\theta\sin\theta$$

$$=0$$

Therefore (10) reduces to

$$(a^2+b^2)\ddot{\theta}=-gb$$

or

$$\ddot{\theta}=-\frac{gb}{(a^2+b^2)}$$

Module No. 183

Example 4 of Equation of Motion in Rotating Frame of Reference

Problem Statement

Prove that the centrifugal force acting on a particle of mass m on the earth's surface is the vector directed away from the earth's center and perpendicular to the angular velocity vector $\vec{\omega}$.

i. Show that its magnitude is $m\omega^2 r_e \cos\lambda$.

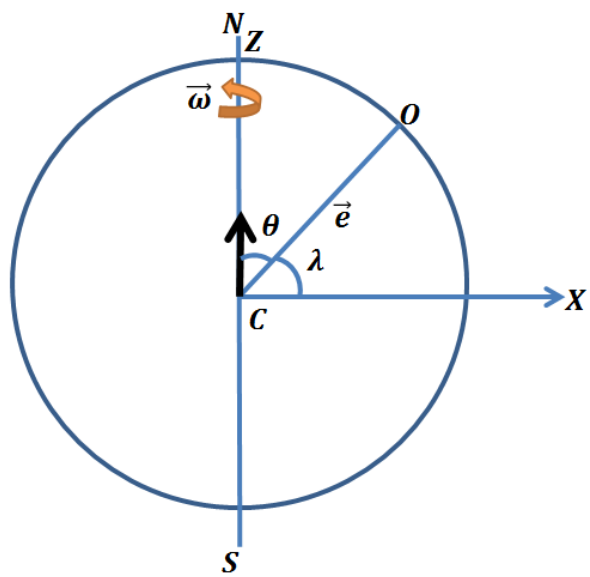
ii. Determine the places on the surface of the earth where centrifugal force will be maximum or minimum.

Solution

i. Since centrifugal force is $-m\vec{\omega} \times (\vec{\omega} \times \vec{r}_e)$.

We have to show that it's vector is directed away from center of the earth and perpendicular to the angular velocity vector $\vec{\omega}$ and its magnitude is $m\omega^2 r_e \cos\lambda$

where λ is the latitude as shown in figure and r_e is the radius of the earth.



It is quite clear that the particle is moving away from the center of earth and centrifugal force is produced in the opposite direction of the path of the particle.

It can be observed from the given figure that the movement of the particle (i.e. $m\vec{\omega} \times (\vec{\omega} \times r_e)$) is perpendicular to the angular velocity $\vec{\omega}$ and vector $\vec{\omega} \times r_e$, becomes centrifugal force in the opposite direction of the particle, so its centrifugal force is also perpendicular to $\vec{\omega}$ and $\vec{\omega} \times r_e$.

Now we move onto the proof of magnitude of centrifugal force, since

$$\vec{F}_{cf} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r}_e)$$

We need to show that

$$\left|\vec{F}_{cf}\right| = |-m\vec{\omega} \times (\vec{\omega} \times \vec{r}_e)| = m\omega^2 r_e \cos\lambda$$

Now we assume that the motion takes place in the XZ-plane.

Then,

$$\hat{e} = \hat{k}\cos\theta + \hat{\imath}\sin\theta$$

$$= \hat{k}\cos\left(\pi/2 - \lambda\right) + \hat{\imath}\sin\left(\pi/2 - \lambda\right)$$

$$\hat{e} = \hat{k}\sin\lambda + \hat{\imath}\cos\lambda$$

Consider the centrifugal force

$$= -m\vec{\omega} \times (\vec{\omega} \times \vec{r}_e)$$

If $\hat{e}$ denotes a unit vector along the NS-axis, then we can write $\vec{\omega} = \omega\hat{e}$ and the expression becomes $\vec{\omega} \times (\vec{\omega} \times \vec{r}_e) = \omega^2\hat{e} \times (\hat{e} \times \vec{r}_e)$

$$-\omega^2\hat{e} \times (\hat{e} \times \vec{r}_e) = -\omega^2[(\hat{e}.\vec{r}_e)\hat{e} - (\hat{e}.\hat{e})\vec{r}_e]$$

$$= -\omega^2[(\hat{e}.\vec{r}_e)\hat{e} - \vec{r}_e]$$

Now

$$\hat{e}.\vec{r}_e = \left(\hat{k}\sin\lambda + \hat{\imath}\cos\lambda\right).r_e\hat{k}$$

$$= r_e\cos\theta = \cos\left(\pi/2 - \lambda\right) = r_e\sin\lambda$$

So expression (1) becomes

$$-\vec{\omega} \times (\vec{\omega} \times \vec{r}_e) = -\omega^2 r_e\sin\lambda\left(\hat{k}\sin\lambda + \hat{\imath}\cos\lambda\right) + \omega^2 r_e\hat{k}$$

$$= (-\omega^2 r_e\sin^2\lambda + \omega^2 r_e)\hat{k} - \omega^2 r_e\sin\lambda\cos\lambda\,\hat{\imath}$$

$$= (-\omega^2 r_e(1 - \cos^2\lambda) + \omega^2 r_e)\hat{k} - \omega^2 r_e\sin\lambda\cos\lambda\,\hat{\imath}$$

$$\left|\vec{F}_{cf}\right| = |-m\vec{\omega} \times (\vec{\omega} \times \vec{r}_e)| = \left|-m((-\omega^2 r_e(1-\cos^2\lambda) + \omega^2 r_e)\hat{k} - \omega^2 r_e \sin\lambda\cos\lambda\,\hat{\imath})\right|$$

$$= \left|-m(\omega^2 r_e \cos^2\lambda)\hat{k} + m\omega^2 r_e \sin\lambda\cos\lambda\,\hat{\imath})\right|$$

$$= m\sqrt{(\omega^2 r_e \cos^2\lambda)^2 + (\omega^2 r_e \sin\lambda\cos\lambda)^2}$$

$$= m\sqrt{\omega^4 {r_e}^2 \cos^4\lambda + \omega^4 {r_e}^2 \cos^2\lambda\sin^2\lambda}$$

$$= m\sqrt{\omega^4 {r_e}^2 \cos^4\lambda + \omega^4 {r_e}^2 \cos^2\lambda\,(1-\cos^2\lambda)}$$

$$= m\sqrt{\omega^4 {r_e}^2 \cos^4\lambda + \omega^4 {r_e}^2 \cos^2\lambda - \omega^4 {r_e}^2 \cos^4\lambda}$$

$$= m\sqrt{\omega^4 {r_e}^2 \cos^2\lambda}$$

$$\left|\vec{F}_{cf}\right| = m\omega^2 r_e \cos\lambda$$

Hence the required result.

The centrifugal force will be maximum at $\lambda = 0,$ where $\cos\lambda = 1$ (maximum) i.e. at equator and minimum will be at NS-poles i.e. at $\lambda = {}^{\pi}/_{2}$ where $\cos\lambda = 0$ (minimum).

Module No. 184

Example 5 of Equation of Motion in Rotating Frame of Reference

Problem Statement

A coordinate system $OXYZ$ is rotating with angular velocity $\vec{\omega} = 5\hat{\imath} - 4\hat{\jmath} - 10\hat{k}$ relative to fixed coordinate system $OX_0Y_0Z_0$ both systems having the same origin. Find the velocity of the particle at rest in the $OXYZ$ system at the point $(3,1,-2)$ as seen by an observer in the fixed system.

Solution

Since the given angular velocity is

$$\vec{\omega} = 5\hat{\imath} - 4\hat{\jmath} - 10\hat{k}$$

and the point P$(3,1,-2)$ at rest in $OXYZ$ system.

Then

$$\overrightarrow{OP} = \vec{r} = 3\hat{\imath} + \hat{\jmath} - 2\hat{k}$$

Also, we know the equation of motion in case when the origins of the coordinate systems coincide each other

$$\overrightarrow{v_f} = \overrightarrow{v_r} + \vec{\omega} \times \vec{r}$$

But we have to find out the velocity of the particle at rest in $OXYZ$ system at P$(3,1,-2)$.

i.e.

$$\overrightarrow{v_f} = \overrightarrow{v_0} + \vec{\omega} \times \vec{r}$$

$$\overrightarrow{v_f} = (0,0,0) + (5\hat{\imath} - 4\hat{\jmath} - 10\hat{k}) \times (3\hat{\imath} + 1\hat{\jmath} - 2\hat{k})$$

$$= \left(0\hat{\imath} + 0\hat{\jmath} + 0\hat{k}\right) + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 5 & -4 & -10 \\ 3 & 1 & -2 \end{vmatrix}$$

$$= \left(0\hat{\imath} + 0\hat{\jmath} + 0\hat{k}\right) + \left((8 + 10)\hat{\imath} - (-10 + 30)\hat{\jmath} + (5 + 12)\hat{k}\right)$$

$$= \left(0\hat{\imath} + 0\hat{\jmath} + 0\hat{k}\right) + \left(18\hat{\imath} - 20\hat{\jmath} + 17\hat{k}\right)$$

Hence

$$\overrightarrow{v_f} = 18\hat{\imath} - 20\hat{\jmath} + 17\hat{k}$$

is the required velocity of the system.

Module No. 185

General Motion of a Rigid Body

In general motion of the body, (i.e. no point of the body is fixed in space), let $\boldsymbol{F}^{ext}$ be the total external force on the rigid body and G_c^{ext} the total external torque about its center of mass (i.e. centroid). Then the equations of motion are

$$M\boldsymbol{a}_c = \boldsymbol{F}^{ext} \qquad (1)$$

and

$$\dot{\boldsymbol{L}}_c = \boldsymbol{G}_c^{ext} \qquad (2)$$

where $\boldsymbol{a}_c$ is the acceleration of the c.m. and $\boldsymbol{L}_c$ is the total angular momentum about it.

Now we resolve the vectors $\boldsymbol{a}_c$,$\boldsymbol{F}^{ext}$,$\boldsymbol{G}_c^{ext}$and $\boldsymbol{L}$ along the unit vector $i,\ j,\ k$ taken along the principal axes at the mass center.

The triad of vectors $i,\ j,\ k$ may be inferred to as a principal triad. It will be assumed to be permanently a principal triad.

Let $\vec{\Omega}$ be its angular velocity. If the triad is fixed in the body then $\vec{\Omega} = \vec{\omega}$, the angular velocity of the body.

Now using the operator: $\left(\frac{d}{dt}\right)_f = \left(\frac{d}{dt}\right)_r + \omega \times$

$$\left(\frac{d\vec{F}}{dt}\right)_f = \left(\frac{d\vec{F}}{dt}\right)_r + \vec{\omega} \times \vec{F}$$

which relates the rate of change of a vector in a fixed (i.e. inertial) frame and a rotating frame, we have (on dropping the suffix r)

$$a_f \equiv \left(\frac{d\vec{v}}{dt}\right)_f = \frac{d\vec{v}}{dt} + \vec{\Omega} \times \vec{v} \qquad (\therefore \boldsymbol{v}_r = \boldsymbol{v})$$

or $\quad a_f = \frac{dv}{dt} + \vec{\Omega} \text{ x } \vec{v} \qquad (3)$

where $v = v_1 i + v_2 j + v_3 k$ is the velocity of the mass center (in the rotating coordinate system). Substituting for $a_f = a_c$ from equation (3) into (1), we obtain

$$M(\frac{d\boldsymbol{v}}{dt} + \vec{\Omega} \times \boldsymbol{v}) = \boldsymbol{F}^{ext}$$

which is equivalent to

$$\left.\begin{array}{l} M\left(\dot{v}_1 + \Omega_2 v_3 + \Omega_3 v_2\right) = F_1 \\ M(\dot{v}_2 + \Omega_3 v_1 + \Omega_1 v_3) = F_2 \\ M(\dot{v}_3 + \Omega_1 v_2 + \Omega_2 v_1) = F_3 \end{array}\right\} \quad (4)$$

From (2), on using

$(\frac{dL}{dt})_f = \frac{dL}{dt} + \Omega \text{ x } L$

and the relation $L = I_1\omega_1\hat{\imath} + I_2\omega_2\hat{\jmath} + I_3\omega_3\hat{k}$, we obtain the equations

$$I_1\dot{\omega}_1\hat{\imath} + I_2\dot{\omega}_2\hat{\jmath} + I_3\dot{\omega}_3\hat{k} + \left(\Omega_2 L_3 - \Omega_3 L_2\right)\hat{\imath} + \left(\Omega_2 L_1 - \Omega_1 L_3\right)\hat{\jmath} + \left(\Omega_1 L_2 - \Omega_2 L_1\right)\hat{k} = \vec{G}$$

From this vector equation we obtain the following three scalar equations

$$I_1\omega_1 + \Omega_2 L_3 - \Omega_3 L_2 = G_1$$

$$I_2\omega_2 + \Omega_3 L_1 - \Omega_1 L_3 = G_2$$

and

$$I_3\omega_3 + \Omega_1 L_2 - \Omega_2 L_1 = G_3$$

where on using the results

$L_1 = I_1\omega_1,\ L_2 = I_2\omega_2,\ L_3 = I_3\omega_3$

we have

$$\left.\begin{array}{l} I_1\omega_1 + \omega_3\Omega_2\mathrm{I}_3 - \omega_2\Omega_3\mathrm{I}_2 = G_1 \\ I_2\omega_2 + \omega_1\Omega_3\mathrm{I}_1 - \omega_3\Omega_1\mathrm{I}_3 = G_2 \\ I_3\omega_3 + \omega_2\Omega_1\mathrm{I}_2 - \omega_1\Omega_2\mathrm{I}_1 = G_3 \end{array}\right\} \quad (5)$$

where $I_1,\ I_2,\ I_3$ denote principal M.I at the centroid of the body.

The set of eqs (4) and (5) constitute six equations for the components of velocity of centroid and the components of angular velocity of the body.

Module No. 186

Equation of Motion Relative to Coordinate System Fixed on Earth

We derived the equation of motion when the origins of fixed and rotating coordinate systems are distant.

$$\vec{a}_f = \vec{a}_0 + \vec{a}_r + 2\vec{\omega} \times \vec{v}_r + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \qquad (1)$$

From (1) the most general from of the equation of motion in a moving frame can be written as

$$m\left(\frac{d^2\vec{r}}{dt^2}\right)_r = \vec{F} - m\dot{\vec{\omega}} \times \vec{r} - 2m\vec{\omega} \times v_r - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - m\vec{a}_0 \qquad (2)$$

where the subscript r refers to the rotating frame, $\vec{F}$ is the total external force in the fixed coordinate system and $\vec{a}_0$ is the acceleration of the origin O in moving (rotating) coordinate system, $OXYZ$. In the case of the earth, which is a rotating coordinate system, we choose the origin as a point fixed on the earth.

We choose the fixed coordinate system at the center C of the earth and denote it by $CX_0Y_0Z_0$. For a particle near the surface of the earth

$$\vec{F} = m\vec{g} \qquad (3)$$

where $\vec{g}$ is the acceleration due to gravity.

$\vec{a}_0$, the acceleration of the origin O is the centripetal acceleration of O due to the rotation of the earth. It may be represented as

$$\vec{a}_0 = \vec{\omega} \times (\vec{\omega} \times \vec{r}_e) \qquad (4)$$

Therefore on substitution from (3) and (4) into (2), we obtain

$$m\left(\frac{d^2\vec{r}}{dt^2}\right)_r = m\vec{g} - m\dot{\vec{\omega}} \times \vec{r} - 2m\vec{\omega} \times v_r - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - m\vec{\omega} \times (\vec{\omega} \times \vec{r}_e) \qquad (5)$$

which gives the full equation of motion for a particle of mass m w.r.t. a coordinate system fixed on the earth.

Next we obtain a simple form of (5) taking into account the fact that the angular velocity $\vec{\omega}$ of the earth is nearly constant both in magnitude and direction, (which is along NS i.e. North-South polar line), and of small magnitude. In fact

$$\omega = |\vec{\omega}| = \frac{2\pi \text{ radian}}{24 \text{ hours}}$$

$$= \frac{2\pi}{86400} \text{ radian per second}$$

$$= 7.27 \times 10^{-5} \text{rad/sec}$$

Since $\vec{\omega}$ may be taken as constant, $\dot{\vec{\omega}} = 0$. Since the fourth term on R.H.S. of (5) has the magnitude $m\omega^2$r (where r is the distance of the particle from the NS-axis), is negligibly small because of ω^2. However the fifth term $m\vec{\omega} \times (\vec{\omega} \times \vec{r}_e)$, the last term of equation (5) is not negligible because $|\vec{r}_e|$ (the magnitude of the radius of the earth) is very large.

Hence equation (5) can be written as

$$m\left(\frac{d^2\vec{r}}{dt^2}\right)_r = m\vec{g} - 2m\vec{\omega} \times v_r - m\vec{\omega} \times (\vec{\omega} \times \vec{r}_e)$$

which is a second order differential equation in $\overrightarrow{r}$ and may also written as

$$m\frac{d^2\vec{r}}{dt^2} + 2m\vec{\omega} \times \frac{d\vec{r}}{dt} = m\vec{g} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}_e) \quad (6)$$

where we have dropped the subscript r, it being understood that the terms on L.H.S of (6) refer to the rotating coordinate system.

If $\hat{e}$ denotes a unit vector along the NS-axis, then we can write $\vec{\omega} = \omega\hat{e},$ and equation of motion takes the form

$$\ddot{\vec{r}} + 2\omega\hat{e} \times \vec{r} = \vec{g} - \omega^2\hat{e} \times (\hat{e} \times \vec{r}_e)$$