Basic Algebra and Trigonometry
MTH 102

Virtual University of Pakistan

Knowledge beyond the boundaries

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Lecture 1-3

Sets and Numbering Systems

The study of Mathematics begins with a study of sets and the development of the numbering systems. Every mathematical system can be represented as a "set"; therefore, it is important for us to understand the definitions, notations and properties of "sets"

- Definition: A **set** is an *unordered* collection of *distinct* objects. Objects in the collection are called **elements** of the set.
- Examples:
    - The collection of persons living in Lahore is a set.
        - Each person living in Lahore is an element of the set.
    - The collection of all towns in the Punjab province is a set.
        - Each town in Punjab is an element of the set.
    - The collection of all quadrupeds is a set.
        - Each quadruped is an element of the set.
    - The collection of all four-legged dogs is a set.
        - Each four-legged dog is an element of the set.
    - The collection of counting numbers is a set.
        - Each counting number is an element of the set.
    - The collection of pencils in your bag is a set.
        - Each pencil in your bag is an element of the set.
- Notation: Sets are usually designated with capital letters. Elements of a set are usually designated with lower case letters.
    - D is the set of all four legged dogs.
    - An individual dog might then be designated by d.
- The **roster method** of specifying a set consists of surrounding the collection of elements with braces. For example the set of counting numbers from 1 to 5 would be written as $\{1, 2, 3, 4, 5\}$.
- **Set builder** notation has the general form {variable | descriptive statement }.

  The vertical bar (in set builder notation) is always read as "such that".

Set builder notation is frequently used when the roster method is either inappropriate or inadequate.

For example, $\{x \mid x < 6$ and x is a counting number$\}$ is the set of all counting numbers less than 6. Note this is the same set as $\{1,2,3,4,5\}$.

- Other Notation: If *x is an element of the set A*, we write this as $x \in A$. $x \notin A$ means *x is not an element of A.*

  If $A = \{3, 17, 2\}$ then $3 \in A$, $17 \in A$, $2 \in A$ and $5 \notin A$.
  If $A = \{x \mid x$ is a prime number $\}$ then $5 \in A$, and $6 \notin A$.

- Definition: The set with no elements is called the **empty set** or the **null set** and is designated with the symbol $\varnothing$.

- Definition: The **universal set** is the set of all things pertinent to a given discussion and is designated by the symbol *U*

  For example, when dealing with all the students enrolled at the Virtual University, the Universal set would be
  U = {all students at the Virtual University}
  Some sets living in this universal set are:
  A = {all Computer Technology students}
  B = {first year students}
  C = {second year students}

- Definition: The set A is a **subset** of the set B, denoted $A \subseteq B$, if every element of A is an element of B.
  If A is a subset of B and B contains elements which are not in A, then A is a **proper subset** of B. It is denoted by $A \subset B$.

  If A is not a subset of B we write $A \not\subset B$ to designate that relationship.

- Definition: Two sets A and B are **equal** if $A \subseteq B$ and $B \subseteq A$. If two sets A and B are equal we write $A = B$ to designate that relationship.
  In other words, two sets, A and B, are equal if they contain the same elements

- Definition: The **intersection** of two sets A and B is the set containing those elements which are elements of A **and** elements of B. We write $A \cap B$ to denote A Intersection B.

- Example: If $A = \{3, 4, 6, 8\}$ and $B = \{1, 2, 3, 5, 6\}$ then $A \cap B = \{3, 6\}$

- Definition: The **union** of two sets A and B is the set containing those elements which are elements of A **or** elements of B. We write $A \cup B$ to denote A Union B.

- Example: If $A = \{3, 4, 6\}$ and $B = \{1, 2, 3, 5, 6\}$ then $A \cup B = \{1, 2, 3, 4, 5, 6\}$.

- Algebraic Properties of Sets:

- Union and intersection are **commutative** operations. In other words, $\mathbf{A \cup B = B \cup A}$ and $\mathbf{A \cap B = B \cap A}$
- Union and intersection are **associative** operations. In other words, $\mathbf{(A \cup B) \cup C = A \cup (B \cup C)}$ and $\mathbf{(A \cap B) \cap C = B \cap (A \cap C)}$
  Union and Intersection are **distributive** with respect to each other. In other words $\mathbf{A \cap (B \cup C) = (A \cap B) \cup (A \cap C)}$ and $\mathbf{A \cup (B \cap C) = (A \cup B) \cap (A \cup C)}$

- A few other elementary properties of intersection and union.
  $\mathbf{A \cup \varnothing = A,\ A \cap \varnothing = \varnothing,\ A \cup A = A,\ A \cap A = A.}$

- Numbering Systems:

- Counting numbers are called **Natural numbers** and the set of Natural numbers is denoted by **N = {**1,2,3…}
- **Integers** are Natural numbers, their opposites and zero. The set of integers is denoted by **Z = {**…-3, -2, -1, 0,1,2,3…}
- **Rational numbers** such as $^2/_3$, $-3^1/_2$ 0.3333, are numbers that can be written as a ratio of two integers. The set of rational numbers is denoted by **Q.** This set includes
  - Repeating decimals, terminating decimals and fractions
  - Integers are also rational numbers since every integer *a* can be written as a fraction *a/1*
- **Irrational numbers** are numbers that can't be written as fractions.
  - 3. 45455455545555… has a pattern but doesn't repeat. It isn't rational. It can't be written like a fraction.
  - Square root of 2, π (Pi) and *e* are also irrational.
- The Union of the set of rational numbers and the set of irrational numbers is the set of **Real numbers,** denoted by **R**
- $\mathbf{N \subset Z \subset Q \subset R}$

- Definition: **Cardinality** refers to the number of elements in a set
  - A *finite* set has a countable number of elements
  - An *infinite* set has at least as many elements as the set of *natural numbers*

- Notation: |A| represents the *cardinality* of Set *A*

- History: Initially numbers were used for counting and the natural numbers did that job well. However there were no solutions for equations of the form $x + 4 = 0$.

  To resolve this, the natural numbers were extended by inventing the negative integers. This was done by attaching a symbol "-" (which we now call the minus sign) to each natural number and calling the new number the "negative" of the original number. This was further extended to all real numbers.

  Now people had solutions for equations of the form $x + 4 = 0$, but equations of the form $x^2 + 4 = 0$ still had no solutions. There is no real number whose square is -4.

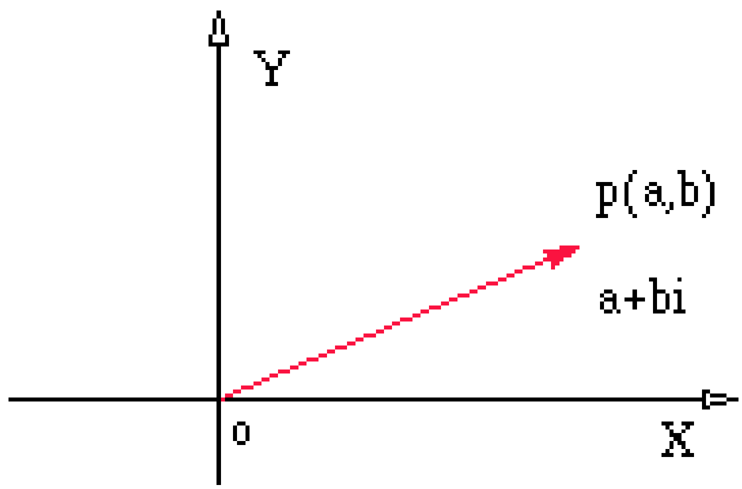
  The numbering system had to be extended once again to accommodate for square roots of negative numbers. A symbol, $= \sqrt{-1}$, was invented and it was called the "imaginary unit". The real numbers were extended by attaching this imaginary unit to each number and calling it the "imaginary copy" of the real numbers.

- Definition: Numbers of the form $a + bi$ are called **complex numbers.**
  - a is the **real** part
  - b is the **imaginary** part

  The set of complex numbers is denoted by **C**

- Examples : 2 - 4i , -3 + 5i and -5 + $^3/_4$i are all complex numbers

- Graphical Representation: Recall that real numbers are represented on a **line.** A complex number has a representation in a **plane**. Simply take the x-axis as the real numbers and y-axis as the imaginary numbers. Thus, giving the complex number $a + bi$ the representation as point P with coordinates (a,b) as the following diagram shows:



- Properties of Complex Numbers:
  - Addition and Subtraction: For complex numbers $a + bi$ and $c + di$,
    $$(a+bi)+(c+di)=(a+c)+(b+d)i$$
    $$(a+bi)-(c+di)=(a-c)+(b-d)i$$
  - Examples
    - $(4 - 6i) + (-3 + 7i) = [4 + (-3)] + [-6 + 7]i = 1 + i$
    - $(10 - 4i) - (5 - 2i) = (10 - 5) + [-4 - (-2)]i = 5 - 2i$
  - Multiplication of Complex Numbers: For complex numbers $a + bi$ and $c + di$,
    $$(a+bi)(c+di)=(ac-bd)+(ad+bc)i.$$
  - The product of two complex numbers is found by multiplying as if the numbers were binomials and using the fact that $i^2 = -1$.

  - Example: $(2 - 4i)(3 + 5i)$

$$=2(3)+2(5i)-4i(3)-4i(5i)$$
$$=6+10i-12i-20i^2$$
$$=6-2i-20(-1)$$
$$=26-2i$$

- Definition: Given a complex number $z = a + ib$, its **conjugate** is defined as $z^* = a - ib$

- Properties of conjugates:
  - $(z^*)^* = z$
  - $z.z^* = a^2 + b^2$ (real)
  - $z + z^* = 2a$ (real)
  - $z - z^* = 2ib$ (imaginary)

- Division of Complex Numbers: For complex numbers $a + bi$ and $c + di$,

$$\frac{(a+bi)}{(c+di)} = \frac{(a+bi)}{(c+di)} x \frac{(c-di)}{(c-di)}$$

$$= \frac{(ac+bd)}{c^2+d^2} + i\frac{(bc-ad)}{c^2+d^2}$$

The quotient of two complex numbers is found by multiplying and dividing by the conjugate of the denominator.

- $$Example: \frac{3+11i}{-1-2i} * \frac{-1+2i}{-1+2i} = \frac{(3+11i)(-1+2i)}{(-1-2i)(-1+2i)} = \frac{-3+6i-11i+22i^2}{1-2i+2i-4i^2} = \frac{-3-5i+22(-1)}{1-4(-1)}$$

$$= \frac{-3-5i-22}{1+4} = \frac{-25-5i}{5} = \frac{-25}{5} - \frac{5i}{5} = -5-i$$

- Definition: The **absolute value** or **modulus** of a complex number is the distance the complex number is from the origin on the complex plane.

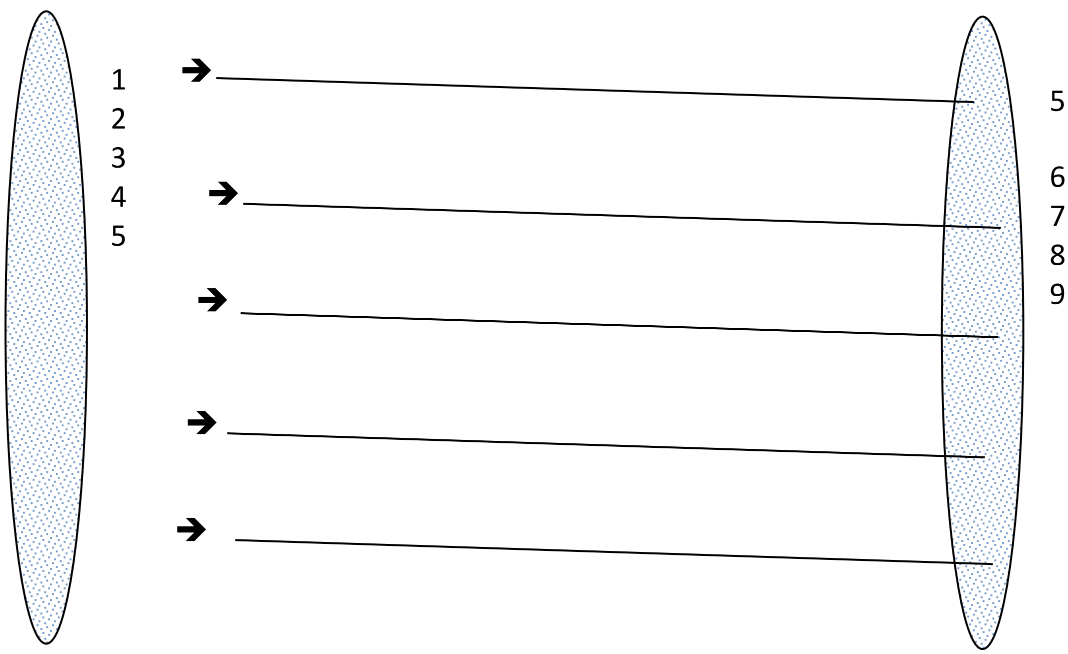
- If you have a complex number $z = (a+bi)$ the absolute value can be found using $|z| = \sqrt{a^2+b^2}$

- Example: $|-2+5i| = \sqrt{(-2)^2+(5)^2} = \sqrt{4+25} = \sqrt{29}$

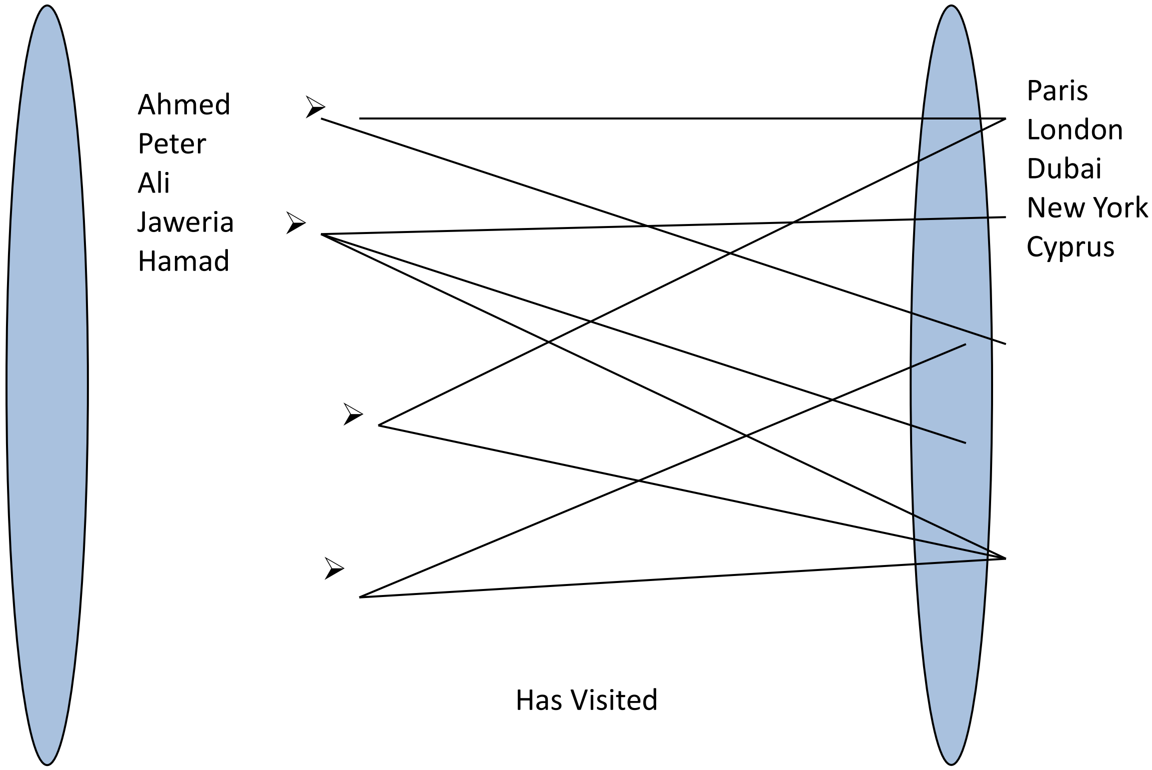
Lecture 4-7

Functions and Quadratics

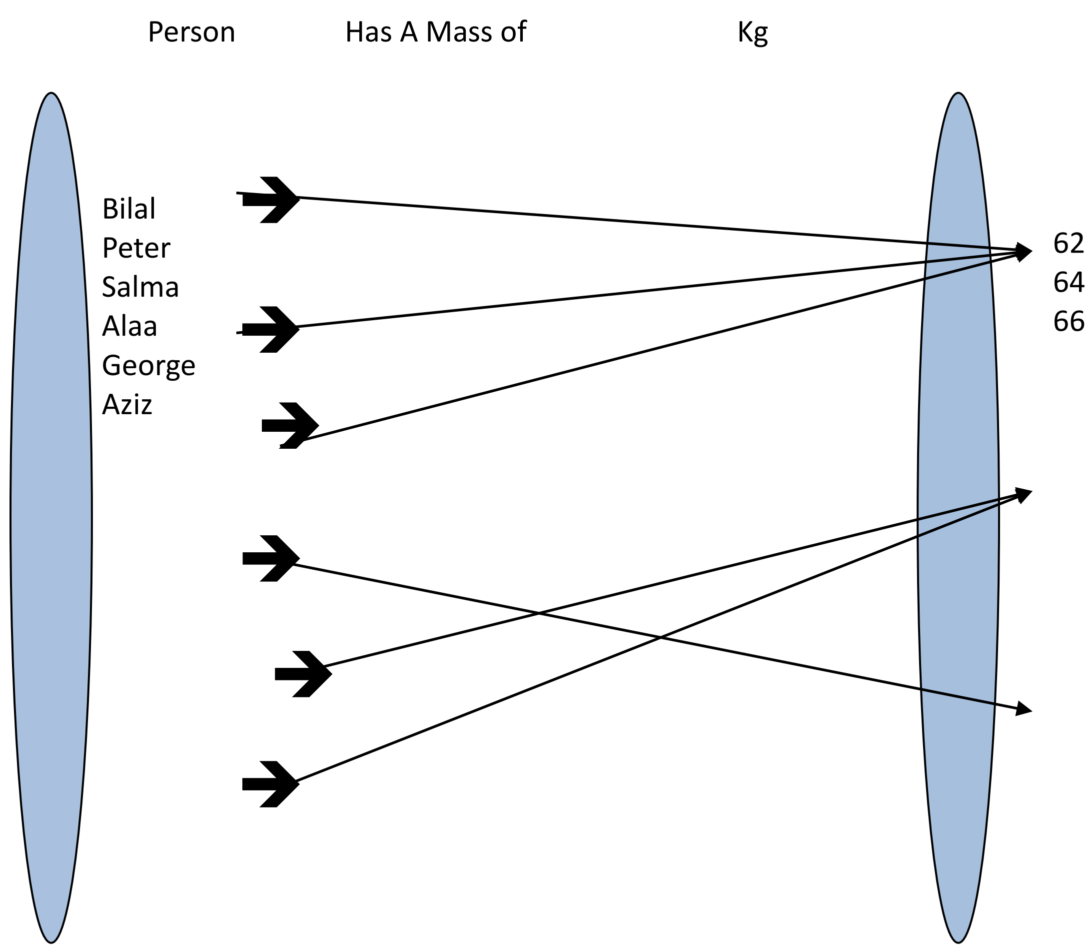
- Definition: A **mapping** between two sets A and B is simply a rule for relating elements of one set to the other. A mapping is also called a **relation**.
- Types of Relations:
  - **One-One Relations** are mappings where each member of the pre-image is mapped to exactly one member of the image.



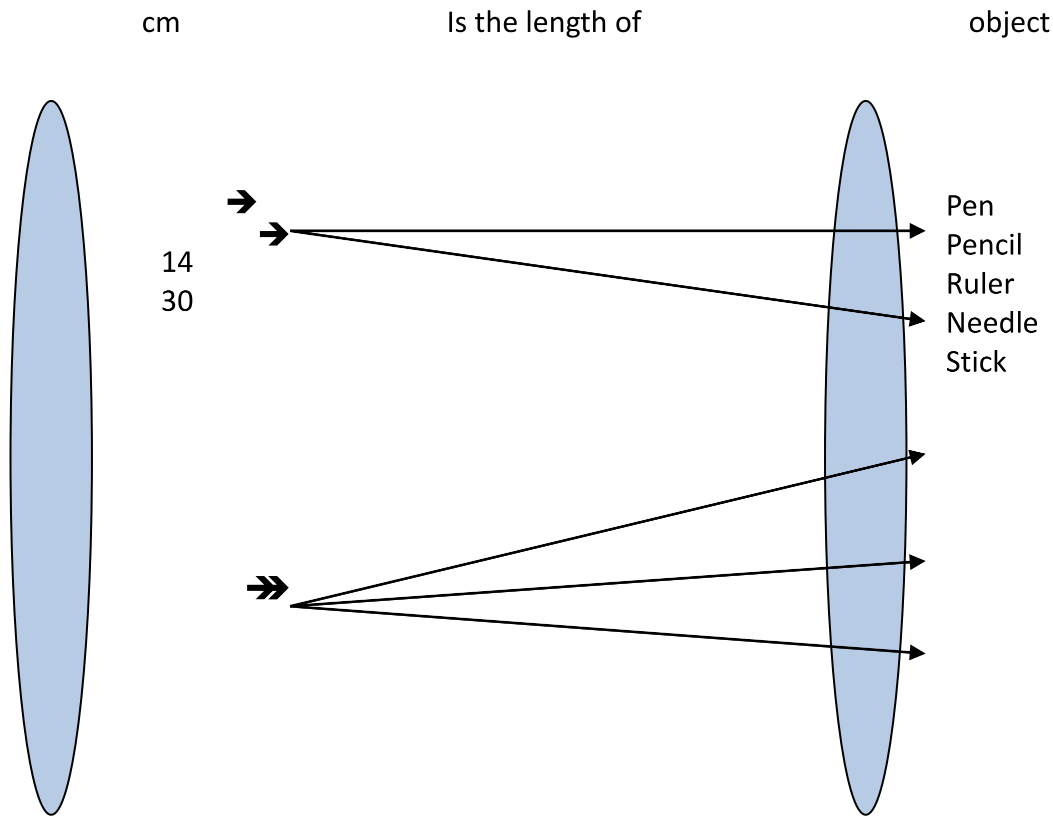
  - **Many – Many Relations** are the mappings where many members of the image are images of more than one member of the pre-image, and members of the pre-image are mapped to more than one image.



- **Many - One Relations** are the mappings where two or more members of the pre-image are mapped to exactly one member of the image.



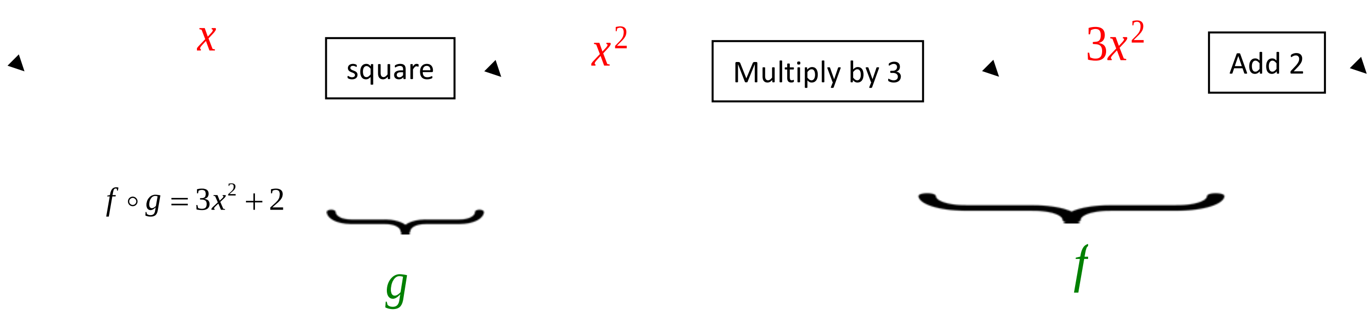
- **One-Many Relations** are mappings where one member of the pre-image is mapped to two or more members of the image.



- Definition: Many-One and One-One relationships are called **functions**.
- Definition: The set consisting of members of the pre-image or inputs of a function is called its **domain.** For a given domain the set of possible outcomes or images of a function is called its **range**.
    - It is important to note that to define a function we need two things: One, the formula for the function, and two, the domain.

- Examples:
    - $f(x) = 2x + 1$; Domain: x = R, Range: $f(x)$ = R
    - $g(x) = \dfrac{1}{x-2}$; Domain: x = R – {2}, Range: $g(x)$ = R
    - $h(x) = \sqrt{x-3}$; Domain: $\{x \in R \mid x \geq 3\}$, Range: $h(x) \geq 0$
    - $q(x) = (x+1)^2 + 2$; Domain: x = R, Range: $q(x) \geq 2$
- Definition: A function is called an **even function** if its graph is symmetric with respect to the vertical axis, and it is called an **odd function** if its graph is symmetric with respect to the origin.

- **Theorem:**
  - If $f(-x) = f(x)$, then f is an even function
  - If $f(-x) = -f(x)$, then f is an odd function
- Example: $f(x) = x^2$ is an even function
- Example: $f(x) = x^3$ is an odd function
- Definition: The **sum**, **difference**, **product** and **quotient** of the functions f and g are the functions defined by
  - $(f + g)(x) = f(x) + g(x)$
  - $(f - g)(x) = f(x) - g(x)$
  - $(fg)(x) = f(x)g(x)$
  - $(f/g)(x) = f(x)/g(x)$, provided $g(x) \neq 0$
- Definition: Given functions f and g, then the function fog is a **composite function**, where g is performed first and then f is performed on the result of g.
- Example: Consider 2 function s $f(x) = 3x + 2$ and $g(x) = x^2$. The function fog may be found using a flow diagram as follows:



$f \circ g = 3x^2 + 2$

Thus

- Remember, the domain of fog is the set of all real numbers x in the domain of g where $g(x)$ is in the domain of f.

The domain of fog cannot always be determined simply by examining the final form of $(fog)(x)$. Any numbers that are excluded from the domain of g must also be excluded from the domain of fog.

- Example: Given $f(x) = \sqrt{4 - x^2}$ and $g(x) = \sqrt{3 - x}$ find $(f \circ g)(x)$ and its domain.

  Solution: Now Domain of f: $-2 \le x \le 2$ and Domain of g: $x \le 3$.

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{3 - x})$$

$$= \sqrt{4 - (\sqrt{3 - x})^2} = \sqrt{4 - (3 - x)} = \sqrt{1 + x}$$

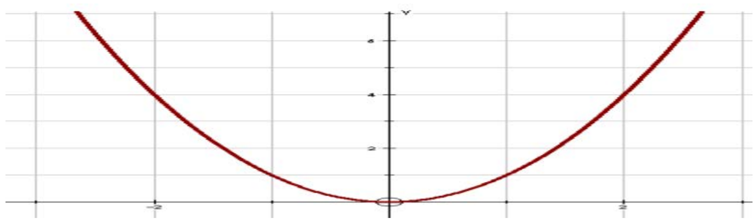
$\sqrt{1 + x}$

Even though is defined for all $x \geq -1$, we must restrict the domain of *fog* to those values that are also in the domain of *g*. Thus, Domain *fog*: $-1 \leq x \leq 3$

- Definition: If *f* is a one-one function, then the **inverse** of *f*, denoted by f^{-1}, is the function obtained by reversing the order of *f*. In other words, if $f(a) = b$ then $f^{-1}(b) = a$.

- If a function is to have an inverse which is also a function then it must be **one-one**. This means that a horizontal line will never cut the graph more than once; i.e. we cannot have $f(a) = f(b)$ if $a \neq b$. Two different inputs (*x* values) are not allowed to give the same output (*y* value).

- Example**:** $f(x) = x^2$ with domain $x \in \Re$ is not one to one. So, for example, the inverse of 4 would have two possibilities: -2 or 2. This means that the inverse is not a function. We say that the inverse function of *f* does not exist. However, if the Domain is restricted to $x \geq 0$, then the function would be one to one and its inverse would be $f^{-1}(x) = \sqrt{x}$, $x \geq 0$

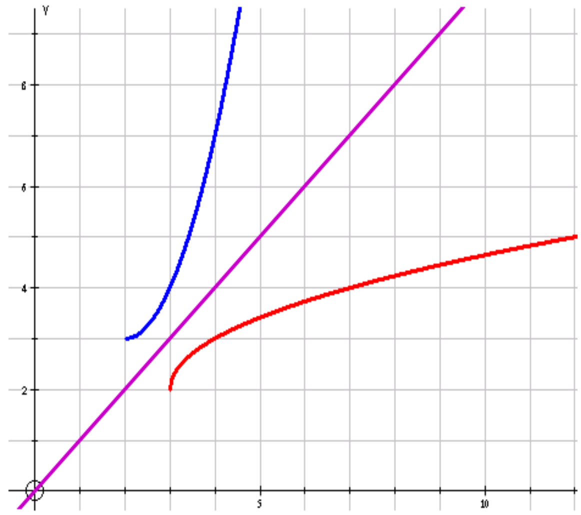


- Properties of Inverse Functions:
    - Domain of the inverse is equal to the Range of *f*.
    - Range of inverse is equal to the Domain of *f*.

- Steps for finding inverse of a function f:
    - Find the domain of *f* and verify that f is one-to-one. If *f* is not one-to-one, then stop as the inverse does not exist.
    - Solve the equation $y = f(x)$ for *x*. i.e. make *x* the subject.
    - Interchange *x* and *y* in step two. This will give the inverse function in terms of *x*.
    - Find the domain of the inverse function.
    - Check that the inverse function is correct.

- Example**:** Find the inverse of the function $y = f(x) = (x-2)^2 + 3$, $x \geq 2$. Sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same axes showing the relationship between them.
  Solution
    - **Step 1:** In order for the function to be one to one, we must restrict its domain to $x \geq 2$. The Range of *f* is $y \geq 3$ and so the domain of f^{-1} will be $x \geq 3$.
    - **Step 2:** Make x the subject. $y - 3 = (x-2)^2 \Rightarrow \sqrt{(y-3)} = x-2 \Rightarrow x = 2 + \sqrt{(y-3)}$
    - **Step 3:** Interchange *x* and *y* in the above equation to get $y = 2 + \sqrt{(x-3)}$. So Final Answer is: $f^{-1}(x) = 2 + \sqrt{(x-3)}$, $x \geq 3$
    - **Step 4:** Verification: $f[f^{-1}(x)] = f[2 + \sqrt{(x-3)}] = \{[2 + \sqrt{(x-3)}] -2\}^2 + 3 = [\sqrt{(x-3)}]^2 + 3 = (x-3) + 3 = x$. And $f^{-1}[f(x)] = f^{-1}[(x-2)^2 + 3] = 2 + \sqrt{([(x-2)^2 + 3] - 3)} = 2 + \sqrt{(x-2)^2} = 2 + (x-2) = x$

  Graph: Reflect in $y = x$ to get the graph of the inverse function.



- Definition: A function of the type $y = ax^2 + bx + c$ where a, b, and c are called the coefficients, is called a **quadratic function**.
  The graph of a quadratic function will form a **parabola**. Each graph will have either a **maximum** or **minimum** point. There is a **line of symmetry** which will divide the graph into two halves.

- What happens if we change the value of a and c?

  - When a is positive, the graph concaves downward.

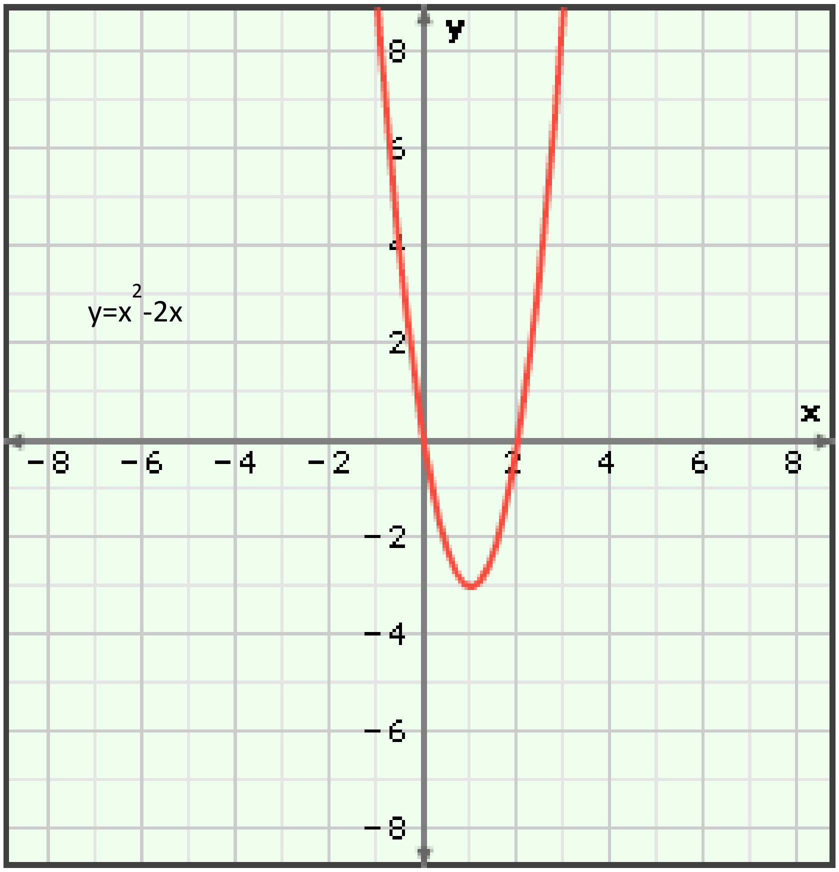
  - When a is negative, the graph concaves upward.

  - When c is positive, the graph moves c units up.

    When c is negative, the graph moves c units down

- Solving Quadratic Equations: Since $y = ax^2 + bx + c$, by setting $y=0$ we set up a quadratic equation. To find the solutions means we need to find the x-intercept. Since the graph is a parabola, there will be at most two solutions.

  - Graphing Method: In this method, we use a scientific calculator and graph the equation. Then we read the x-intercepts from the graph.
  -
    - Example: $x^2 - 2x = 0$
    - To solve the equation, write $y = x^2 - 2x$ into your graphing calculator. Find the x-intercepts. The two solutions are $x=0$ and $x=2$.



- Factorization Method: To solve a quadratic equation we get it in the standard form $y = ax^2 + bx + c$ and see if it will factorize.
  - Example: $x^2 = 5x - 6$ ➔ $x^2 - 5x + 6 = 0$ ➔ $(x-3)(x-2) = 0$
    ➔ $x - 3 = 0$ or $x - 2 = 0$
    ➔ $x = 3$ and $x = 2$

- Completing the Square Method: For this method we need the coefficient of x^2 to be 1. We then divide the take the coefficient of x and add and subtract the square of half of the coefficient of x from the equation to form a perfect square on one side of the equation.
  - Example: $x^2 + 6x + 3 = 0$. This does not factorize. So we will use the completing the square method here.

    $x^2 + 6x = -3$ The coefficient of x is 6. So, the square of 6/2 is 9.
    $x^2 + 6x + \underline{9} = -3 + \underline{9}$. We added 9 to both sides of the equation

    $x^2 + 6x + 9 = 6$. The left side becomes a perfect square.

    $(x+3)^2 = 6$ ➔ $x + 3 = \pm\sqrt{6}$ ➔ $x = -3 \pm \sqrt{6}$

    $x = -3 + \sqrt{6} \approx -0.55$ And $x = -3 - \sqrt{6} \approx -5.45$

- Formula Method: Using the completing the square method, we get the general quadratic formula. Given $ax^2 + bx + c = 0$, the quadratic formula is:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- Example: $x^2 + 6x + 3 = 0$
  Here, $a = 1$, $b = 6$ and $c = 3$. So, using the formula, we get:

  $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-6 \pm \sqrt{6^2 - 4(1)(3)}}{2(1)} = \frac{-6 \pm \sqrt{24}}{2} = -3 \pm \sqrt{6}$$

- Note that the expression $b^2 - 4ac$ under the square root – called the **discriminant** - determines how many solutions (if any) the quadratic equation will have.

  - If $b^2 - 4ac > 0$. there will be two distinct real solutions.
  - If $b^2 - 4ac = 0$ there will be exactly one real solution.
  - If $b^2 - 4ac < 0$ there will be no real solutions.

- Examples**:** In each of the following cases determine if the equations has one, two or zero real solutions.
  - $2x^2 + 7x + 4 = 0$
    $b^2 - 4ac = 7^2 - 4(2)(4) = 49 - 32 = 17 > 0$
    Therefore, there are two distinct real solutions

  - $2x^2 + 4x + 2 = 0$
    $b^2 - 4ac = 4^2 - 4(2)(2) = 16 - 16 = 0$
    Therefore, there is exactly one real solution

  - $3x^2 + 4x + 2 = 0$
    $b^2 - 4ac = 4^2 - 4(3)(2) = 16 - 24 = -8 < 0$
    Therefore, there are no real solutions.

Lecture 8-12

Matrices and Determinants

- Definition: A **matrix** is a rectangular arrangement of numbers in rows and columns. The **order** of a matrix is the number of the rows and columns. The **entries** are the numbers in the matrix.

- Notation: A matrix A is written as follows:

$$A=\begin{bmatrix} a_{11}\ a_{12}\ \ldots a_{1n} \\ a_{21}\ a_{22}\ \ldots a_{2n} \\ \ldots\ldots\ldots\ldots\ldots \\ a_{m1}\ a_{m2}\ \ldots a_{mn} \end{bmatrix}=\left(a_{ij}\right)$$

where a_{rs} represents the entry in the rth row and sth column. A matrix with m rows and n columns is said to have order m x n.

- Operations on Matrices:

  - Addition of Matrices: To add two matrices, they must have the same order. To add, we simply add corresponding entries.
  - Example:

$$\begin{bmatrix} 5 & -3 \\ -3 & 4 \\ 0 & 7 \end{bmatrix}+\begin{bmatrix} -2 & 1 \\ 3 & 0 \\ 4 & -3 \end{bmatrix}=\begin{bmatrix} 5+(-2) & -3+1 \\ -3+3 & 4+0 \\ 0+4 & 7+(-3) \end{bmatrix}=\begin{bmatrix} 3 & -2 \\ 0 & 4 \\ 4 & 4 \end{bmatrix}$$

  - Subtraction of Matrices: To subtract two matrices, they must have the same order. We simply subtract corresponding entries.
  - Example:

$$\begin{bmatrix} 9 & -2 & 4 \\ 5 & 0 & 6 \\ 1 & 3 & 8 \end{bmatrix}-\begin{bmatrix} 4 & 0 & 7 \\ 1 & 5 & -4 \\ -2 & 3 & 2 \end{bmatrix}=\begin{bmatrix} 9-4 & -2-0 & 4-7 \\ 5-1 & 0-5 & 6-(-4) \\ 1-(-2) & 3-3 & 8-2 \end{bmatrix}$$

$$=\begin{bmatrix} 5 & -2 & -3 \\ 4 & -5 & 10 \\ 3 & 0 & 6 \end{bmatrix}$$

- Multiplication by a Scalar: In matrix algebra, a real number is often called a **scalar**. To multiply a matrix by a scalar, we multiply each entry in the matrix by that scalar.
- Example:

$$4\begin{bmatrix} -2 & 0 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 4(-2) & 4(0) \\ 4(4) & 4(-1) \end{bmatrix} = \begin{bmatrix} -8 & 0 \\ 16 & -4 \end{bmatrix}$$

- Matrix Multiplication: Given an r x c matrix A and an s x d matrix B, we can multiply A with B to form the matrix AB if $c = s$. The resulting matrix will have dimension r x d.

  To multiply two matrices, we multiply each row in the first matrix by each column in the second matrix. An example is shown below:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix},$$

  We can multiply to form the matrix $C = AB$ since the number of columns of A is equal to the number of rows of B. The multiplication is carried out as follows:

$$C = AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \end{bmatrix}$$

- Example:

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

$$C = AB = \begin{bmatrix} (2)(1)+(3)(1)=5 & (2)(1)+(3)(0)=2 & (2)(1)+(3)(2)=8 \\ (1)(1)+(1)(1)=2 & (1)(1)+(1)(0)=1 & (1)(1)+(1)(2)=3 \\ (1)(1)+(0)(1)=1 & (1)(1)+(0)(0)=1 & (1)(1)+(0)(2)=1 \end{bmatrix} = \begin{pmatrix} 5 & 2 & 8 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix}$$

- Definition: A Square matrix with ones on the diagonal and zeros elsewhere is called an **identity** matrix. It is denoted by I.

- Example: The 4 x 4 identity matrix is:

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Definition: Given a square matrix B, if there exists a matrix D such that $BD = DB = I$, then D is called the **inverse** of B, and is denoted by $D = B^{-1}$.

- Given a system of equations $\sum_{j=1}^{n} a_{ij}x_j = b_i$, it can be written as a matrix equation $A\underline{x} = \underline{b}$,

  where *A* is the matrix of the coefficients, $\underline{x}$ is the column matrix of the variables and $\underline{b}$ is the column matrix of the constants. i.e.

$$, \quad \underline{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad A = \begin{bmatrix} a_{11}\ a_{12} \ldots a_{1n} \\ a_{21}\ a_{22} \ldots a_{2n} \\ \ldots\ldots\ldots\ldots\ldots \\ a_{m1}\ a_{m2} \ldots a_{mn} \end{bmatrix}$$

  If A^{-1} exists then the system has a unique solution given by $\underline{x} = A^{-1}\underline{b}$

- Definition: A matrix is in **echelon form** if it has the following properties
  - Every non-zero row begins with a 1 (called a **leading** 1)
  - Every leading one in a lower row is further to the right of the leading one above it.
  - If there are zero rows, they are at the end of the matrix

  A matrix is in **reduced echelon form** if in addition to the above three properties it also has the following property:
  - Every other entry in a column containing a leading one is zero

- Methods for finding Solutions of Equations:
  - Using Row Operations: Recall that when we are solving simultaneous equations, the system of equations remains unchanged if we perform the following operations:

    - Multiply an equation by a non-zero constant
    - Add a multiple of one equation to another equation
    - Interchange two equations.

    We have seen that any system of equations can be written as a matrix system. i.e. the two systems are equivalent.
    So, given a system ***Ax = b*** we can form the **augmented matrix** (***Ab***) by attaching an additional column at the end of the matrix A with entries from matrix b. Since the original system of equations remains unchanged as described above, the system described by the augmented matrix (***Ab***) also remains unchanged under the following row operations:

    - Multiply a row by a non-zero constant
    - Add a multiple of one row to another row
    - Interchange two rows.

    Using row operations, we will change the matrix (***Ab***) to an Echelon form or a reduced Echelon form. Once that is achieved, the solution will be easily found.

    Example: Solve the following system of equations:

$$x + 2y + z = 1$$
$$2x + 2y = 2$$
$$3x + 5y + 4z = 1$$

Solution: The system can be written in matrix form as:

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 0 \\ 3 & 5 & 4 \end{bmatrix}\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

The augmented matrix is

$$(\boldsymbol{Ab}) = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 2 & 0 & 2 \\ 3 & 5 & 4 & 1 \end{bmatrix} \underset{\boldsymbol{R}_3-3\boldsymbol{R}_1}{\overset{\boldsymbol{R}_2-2\boldsymbol{R}_1}{\sim}} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & -1 & 1 & -2 \end{bmatrix} \overset{\boldsymbol{R}_{23}}{\sim} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -1 & 1 & -2 \\ 0 & -2 & -2 & 0 \end{bmatrix}$$

$$\overset{-\boldsymbol{R}_2}{\sim} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & -2 & -2 & 0 \end{bmatrix} \underset{\boldsymbol{R}_3+2\boldsymbol{R}_2}{\overset{\boldsymbol{R}_1-2\boldsymbol{R}_2}{\sim}} \begin{bmatrix} 1 & 0 & 3 & -3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & -4 & 4 \end{bmatrix} \overset{-\boldsymbol{R}_3/4}{\sim} \begin{bmatrix} 1 & 0 & 3 & -3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\underset{\boldsymbol{R}_2+\boldsymbol{R}_3}{\overset{\boldsymbol{R}_1-3\boldsymbol{R}_3}{\sim}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

This gives the solution: $x = 0, y = 1, z = -1$

- Solving Equations using Inverse: If we could find A^{-1}, we could also solve the system by using $\underline{x} = A^{-1}\underline{b}$ One way to find inverse is as follows:

- **Theorem**: Given a *nxn* matrix *A*, if the augmented matrix (*AI*), where *I* is the *nxn* identity matrix, can be row reduced to a matrix (*IB*), then *B* is the inverse of *A*. If (*AI*) cannot be reduced to (*IB*), then A does not have an inverse.

- Example: Solve the system of equations:

$$2x - 2y + 2z = 1$$
$$2y - z = 1$$
$$2x + 3y = -1$$

Solution: The matrix *A* of the coefficients is

$$\begin{bmatrix} 2 & -2 & 2 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix}$$

To find A^{-1} we use the above theorem

$$(\boldsymbol{AI}) = \begin{bmatrix} 2 & -2 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 0 & 1 \end{bmatrix} \overset{\boldsymbol{R}_3-\boldsymbol{R}_1}{\sim} \begin{bmatrix} 2 & -2 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 0 & 5 & -2 & -1 & 0 & 1 \end{bmatrix}$$

$$\underset{\boldsymbol{R}_{23}}{\overset{\boldsymbol{R}_1/2}{\sim}} \begin{bmatrix} 1 & -1 & 1 & 1/2 & 0 & 0 \\ 0 & 5 & -2 & -1 & 0 & 1 \\ 0 & 2 & -1 & 0 & 1 & 0 \end{bmatrix} \overset{\boldsymbol{R}_2-2\boldsymbol{R}_3}{\sim} \begin{bmatrix} 1 & -1 & 1 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -1 & -2 & 1 \\ 0 & 2 & -1 & 0 & 1 & 0 \end{bmatrix}$$

$$\underset{\boldsymbol{R}_3-2\boldsymbol{R}_2}{\overset{\boldsymbol{R}_1+\boldsymbol{R}_2}{\sim}} \begin{bmatrix} 1 & 0 & 1 & -1/2 & -2 & 1 \\ 0 & 1 & 0 & -1 & -2 & 1 \\ 0 & 0 & -1 & 2 & 5 & -2 \end{bmatrix} \underset{-\boldsymbol{R}_3}{\overset{\boldsymbol{R}_1+\boldsymbol{R}_3}{\sim}} \begin{bmatrix} 1 & 0 & 0 & 3/2 & 3 & -1 \\ 0 & 1 & 0 & -1 & -2 & 1 \\ 0 & 0 & 1 & -2 & -5 & 2 \end{bmatrix}$$

So,

$$\boldsymbol{A}^{-1} = \begin{bmatrix} 3/2 & 3 & -1 \\ -1 & -2 & 1 \\ -2 & -5 & 2 \end{bmatrix}$$

This gives the solution

$$\underline{\boldsymbol{x}} = \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{pmatrix} = \boldsymbol{A}^{-1}\underline{\boldsymbol{b}} = \begin{bmatrix} 3/2 & 3 & -1 \\ -1 & -2 & 1 \\ -2 & -5 & 2 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3/2+3+1 \\ -1-2-1 \\ -2-5-2 \end{pmatrix} = \begin{pmatrix} 11/2 \\ -4 \\ -9 \end{pmatrix}$$

- Definition: Let M be the set of all square matrices. Then the **Determinant** is a function from M to the set of real numbers. i.e. the determinant is a process of attaching a real number to every square matrix.

- Notation: If A is a square matrix, then determinant of A is denoted by **detA** or **|A|**

- Example**:** $\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

- Second Order Determinant**:** A 2 x 2 determinant can be found using the following method:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{22}$$

- Example: $\begin{vmatrix} -1 & 2 \\ -3 & -4 \end{vmatrix} = (-1)(-4) - (-3)(2) = 4 + 6 = 10$

- Definition: The **Minor of an element** in a third-order determinant is a second-order determinant obtained by deleting the row and column that contains the element.

- Example**:** Given $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

  The Minor of $a_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$

  The Minor of $a_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$

- Definition: The **Cofactor** of $a_{ij} = (-1)^{i+j}$ (Minor of a_{ij})

- The cofactor of a_{ij} is denoted by C_{ij}

- Thus, the cofactor of an element is nothing more than a signed minor.

- Example**:** Given $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

  The Cofactor of $a_{23} = C_{23} = (-1)^{2+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$

  The Cofactor of $a_{11} = C_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

- Theorem: The value of a determinant of order 3 is the sum of three products obtained by multiplying each element of any one row (or each element of any one column) by its cofactors.

- ***Note***: The above theorem and definitions of minor and cofactor generalize completely for determinants of order higher than 3

- Example**:** Evaluate $\begin{vmatrix} 2 & -2 & 0 \\ -3 & 1 & 2 \\ 1 & -3 & -1 \end{vmatrix}$

Solution**:** We expand by the first row to get:

$$\begin{vmatrix} 2 & -2 & 0 \\ -3 & 1 & 2 \\ 1 & -3 & -1 \end{vmatrix} =$$

$$2\left[(-1)^{1+1}\begin{vmatrix} 1 & 2 \\ -3 & -1 \end{vmatrix}\right] + (-2)\left[(-1)^{1+2}\begin{vmatrix} -3 & 2 \\ 1 & -1 \end{vmatrix}\right] + (0)\left[(-1)^{1+1}\begin{vmatrix} -3 & 1 \\ 1 & -3 \end{vmatrix}\right] =$$

$$(2)(1)[(1)(-1)-(-3)(2)] + (-2)(-1)[(-3)(-1)-(1)(2)] + 0 =$$

$$(2)(5)+(2)(1) = 12$$

- Finding Inverse Using Determinants

- Definition: Given a matrix *A*, the **transpose** of *A* denoted A^T is the matrix obtained by interchanging the rows of *A* with its columns.

- Example: The transpose of $A = \begin{bmatrix} 8 & -1 & 3 \\ 0 & 0 & 2 \\ 10 & 4 & -3 \end{bmatrix}$ is $A^T = \begin{bmatrix} 8 & 0 & 10 \\ -1 & 0 & 4 \\ 3 & 2 & -3 \end{bmatrix}$

- Definition: Given a matrix *A*, calculate all the cofactors of *A*. We then form the matrix (C_{ij}) of the cofactors. The **Adjoint** or **Adjugate** of *A* is the transpose of the matrix of the cofactors. i.e. $adj(A) = (C_{ij})^T = (C_{ji})$

- The inverse of A is then found by the formula: $A^{-1} = \dfrac{1}{|A|} adj(A)$

- Example: Find the inverse of the following matrix using determinant.

$$A = \begin{bmatrix} -2 & 1 & 3 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

Solution: We need to first find all the cofactors.

$$C_{11} = (-1)^{1+1}\begin{vmatrix} -1 & 1 \\ 2 & 0 \end{vmatrix} = -2 \qquad C_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 1 \qquad C_{13} = (-1)^{1+3}\begin{vmatrix} 0 & -1 \\ 1 & 2 \end{vmatrix} = 1$$

$$C_{21} = (-1)^{2+1}\begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = 6 \qquad C_{22} = (-1)^{2+2}\begin{vmatrix} -2 & 3 \\ 1 & 0 \end{vmatrix} = -3 \qquad C_{23} = (-1)^{2+3}\begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix} = 5$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} = 4 \qquad C_{32} = (-1)^{3+2}\begin{vmatrix} -2 & 3 \\ 0 & 1 \end{vmatrix} = 2 \qquad C_{33} = (-1)^{3+3}\begin{vmatrix} -2 & 1 \\ 0 & -1 \end{vmatrix} = 2$$

Now, $adj(A)=\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} -2 & 6 & 4 \\ 1 & -3 & 2 \\ 1 & 5 & 2 \end{bmatrix}$. And

$$\det(A)=\begin{vmatrix} -2 & 1 & 3 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{vmatrix} = -2\begin{vmatrix} -1 & 1 \\ 2 & 0 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + 3\begin{vmatrix} 0 & -1 \\ 1 & 2 \end{vmatrix} = -2(-2)-1(-1)+3(1)=8$$

Therefore, $A^{-1}=\dfrac{1}{\det(A)}adj(A)=\dfrac{1}{8}\begin{bmatrix} -2 & 6 & 4 \\ 1 & -3 & 2 \\ 1 & 5 & 2 \end{bmatrix}$

- Cramer's Rule for Solving Systems of Equations: Suppose we are given a system of 2 equations in 2 unknowns as follows:

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

Then, the solution is given by: $x_1 = \dfrac{\Delta_1}{\Delta}$ and $x_2 = \dfrac{\Delta_2}{\Delta}$, where $\Delta = \det(A)$, $\Delta_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}$

Similarly, for a system involving three unknowns and three equations:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_2 = b_3$$

The solution is given by $x_1 = \dfrac{\Delta_1}{\Delta}$, $x_2 = \dfrac{\Delta_2}{\Delta}$ and $x_3 = \dfrac{\Delta_3}{\Delta}$, where $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$,

$\Delta_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$ and $\Delta_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$

- Example: Solve using Cramer's Rule:

$$x_1 + 2x_2 + x_3 = 1$$
$$2x_1 + 2x_2 = 2$$
$$3x_1 + 5x_2 + 4x_3 = 1$$

Solution:

$$\Delta = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 2 & 0 \\ 3 & 5 & 4 \end{vmatrix} = 1\begin{vmatrix} 2 & 0 \\ 5 & 4 \end{vmatrix} - 2\begin{vmatrix} 2 & 0 \\ 3 & 4 \end{vmatrix} + 1\begin{vmatrix} 2 & 2 \\ 3 & 5 \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 5 & 4 \end{vmatrix} = 1\begin{vmatrix} 2 & 0 \\ 5 & 4 \end{vmatrix} - 2\begin{vmatrix} 2 & 0 \\ 1 & 4 \end{vmatrix} + 1\begin{vmatrix} 2 & 2 \\ 1 & 5 \end{vmatrix}$$

$$= 1(8) - 2(8) + 1(4) = 8 - 16 + 4 = -4 \qquad = 1(8) - 2(8) + 1(8) = 8 - 16 + 8 = 0$$

$$, \ \Delta_2 = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 0 \\ 3 & 1 & 4 \end{vmatrix} = 1\begin{vmatrix} 2 & 0 \\ 1 & 4 \end{vmatrix} - 1\begin{vmatrix} 2 & 0 \\ 3 & 4 \end{vmatrix} + 1\begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix}, \text{ and}$$

$$= 1(8) - 1(8) + 1(-4) = 8 - 8 - 4 = -4$$

$$\Delta_3 = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 3 & 5 & 1 \end{vmatrix} = 1\begin{vmatrix} 2 & 2 \\ 5 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 2 \\ 3 & 5 \end{vmatrix}$$

$$= 1(-8) - 2(-4) + 1(4) = -8 + 8 + 4 = 4$$

So, by Cramer's Rule: $x_1 = \dfrac{\Delta_1}{\Delta} = \dfrac{0}{-4} = 0$, $x_2 = \dfrac{\Delta_2}{\Delta} = \dfrac{-4}{-4} = 1$, $x_3 = \dfrac{\Delta_3}{\Delta} = \dfrac{4}{-4} = -1$.

Lecture 13-15

Sequences and Series

- Definition: Rows of numbers are called **sequences**, and the separate numbers are called **terms** of the sequence.

- Notation: Usually the terms of the sequence are denoted by
  a_1 = First term of the sequence
  a_2 = second term of the sequence
  …
  a_n = nth term of the sequence

- Definition: An **Arithmetic Sequence** (or **Arithmetic Progression**) is a sequence in which each term after the first term is found by adding a constant, called the common difference (d), to the previous term.

- The inductive definition of an arithmetic sequence has the form
  $a_1 = a$ and $a_n = a_{n-1} + d$

- Formula: The formula for finding any term in an arithmetic sequence is $\boldsymbol{a_n = a + (n-1)d}$.

  All you need to know to find any term is the first term in the sequence (a) and the common difference, d.

- Example: Given the sequence, 1500, 3000, 4500, 6000, …, find the 16th term.

  Solution: $a = \mathbf{1500}$, $d = \mathbf{1500}$, $n - 1 = 16 - 1 = \mathbf{15}$.

  So $a_{16} = 1500 + 1500(15) = 24000$

- Definition**:** An **Arithmetic Series** is the sum of the terms in an arithmetic sequence.

- Example**:** Arithmetic sequence: 2, 4, 6, 8, 10,

  Corresponding arithmetic series: 2 + 4 + 6 + 8 + 10

- To find the sum of the first n terms on arithmetic sequence, we can use the formula:

  $\boldsymbol{S_n = {}^{n}/_{2}(a + a_n) = {}^{n}/_{2}(a + a + (n\text{-}1)d) = {}^{n}/_{2}(2a + (n\text{-}1)d)}$

- Example: Find the sum of the first 50 terms of an arithmetic series with $a = 28$ and $d = -4$

  Solution: We need to know n, a, and a_{50}. $n= 50$, $a = 28$, $d = -4$. The formula is $S_n = {}^{n}/_{2}(2a + (n\text{-}1)d)$ So, $S_{50} = {}^{50}/_{2}(2(28) + (50\text{-}1)(\text{-}4)) = 25(56 + 49 (-4)) = 25(56 – 196) = 25(-140) = -3500$

- Definition: A sequence in which each term after the first is found by multiplying the previous term by a constant value called the common ratio, is called a **Geometric Sequence** (or **Geometric Progression**).

- The formula for finding any term of a geometric sequence is $a_n = ar^{n-1}$

- Example: Find the 10th term of the geometric sequence with $a = 2000$ and a common ratio of ${}^{1}/_{2}$.
- Solution: $a_{10} = 2000\bullet ({}^{1}/_{2})^9 = 2000 \bullet {}^{1}/_{512} = {}^{2000}/_{512} = {}^{500}/_{128} = {}^{250}/_{64} = \boldsymbol{{}^{125}/_{32}}$

- Definition**:** A **Geometric Series** is the sum of the terms in a arithmetic sequence.

- The formula for fining the sum of the first n terms of a geometric sequence is given by

  $$S_n = \frac{a(1-r^n)}{1-r}$$

- Example: Find the sum $\sum_{n=1}^{4} -3(2)^{n-1}$

  Solution: This is a geometric series with first term -3 and common ratio 2.

  $$S_4 = \frac{-3(1-2^4)}{1-2} = \frac{-3(1-16)}{-1} = -45$$

- Definition: If a sequence of numbers approaches (or converges) to a finite number, we say that the sequence is **convergent.** If a sequence does not converge to a finite number it is called divergent.

- For the sequence of the geometric series, we know that $S_n = \frac{a(1-r^n)}{1-r}$ and it is the expression r^n that determines if the series converges or diverges: If $|r|>1$ then $|r^n|$ increases indefinitely and the series is divergent. However, if $|r|<1$ then $|r^n|$ tends to zero as n tends to infinity. So the sum tends to $S_n = \frac{a(1-0)}{1-r} = \frac{a}{1-r}$. It is conventional to write $S_\infty = \frac{a}{1-r}$

- Example: Evaluate $\sum_{n=1}^{\infty} 35\left(-\frac{1}{4}\right)^{n-1}$

Solution: This is an infinite geometric series with $a = 35$ and $r = -1/4$. So,

$$S_\infty = \frac{35}{1-(-1/4)} = \frac{35}{(5/4)} = 28$$

Lecture 16 – Review Sets

Lecture 17 – Review Functions

Lecture 18 – Review Matrices

Lecture 19 - 20

Permutations and Combinations

- Multiplication Principle: If two operations A and B are performed in order, with *n* possible outcomes for A and *m* possible outcomes for B, then there are n x m possible combined outcomes of the first operation followed by the second.
- Example: Basket A contains a mango (m) and a banana (b). Basket B contains an apple (a), an orange (o) and a grapefruit (g). You draw one fruit from A and then one fruit from B. How many different pairs of fruits can you have?

  Solution: ma, mo, mg, ba, bo, bg = 6. Or, (2 from A) x (3 from B) = 6 according to the multiplication principle.
- Difference between permutations and combinations: Both are ways to count the possibilities. The difference between them is whether order matters or not. Consider a poker hand:

  A♦, 5♥, 7♣, 10♠, K♠

  Is that the same hand as:

  K♠, 10♠, 7♣, 5♥, A♦

  Does the order the cards are handed out matter?

  - If yes, then we are dealing with permutations
  - If no, then we are dealing with combinations
- Permutation Rules:
  - To arrange *n* distinct objects in a row, the number of different arrangements is $n! = n(n-1)(n-2)\ldots3.2.1$
  - The number of different permutations of *r* objects which can be made from *n* distinct objects is given by
  - The number of different permutations of *n* objects of which *p* are identical to each other, and then *q* of the remainder are identical, and *r* of the remainder are identical is
  - To arrange *n* objects in a line in which *r* of the objects have to be together we have $r!(n-r+1)!$ Permutations
  - To arranging *n* objects in a circle in which arrangements are considered to be the same if they can be obtained from each other by rotation, we get $n!/n = (n-1)!$ Permutations.

- Example: Suppose you have 4 pictured cards that have the pictures of the letters A, B, C and D, and you want to arrange them in a row to form "words". How many 4-letter words are there?

  Solution: here we are arranging four distinct objects in a line. The number of permutations is 4! = 24

- Example: Eight runners are hoping to take part in a race, but the track has only six lanes. In how many ways can six of the eight runners be assigned to lanes.

  Solution: This is a permutation of six lane assignments from 8 people:
  $$^{8}P_{6} = \frac{8!}{(8-6)!} = \frac{8!}{2!} = \frac{8.7.6.5.4.3.2.1}{2.1} = 20160$$

- Example: Find the number of distinct permutations of the letters of the word MISSISSIPPI

  Solution: The total number of letters = 11, Number of S's = 4, Number of I's = 4, Number of P's = 2. The total number of distinct permutations is $\frac{11!}{4!4!2!} = 34650$

- Example: Five people, A, B, C, D and E are arranged randomly in a line. Find the possible permutations when A and B are next to each other

  Solution: Imagine A and B are stuck together in the order AB. Treat them as one unit. Then there are 4 unites to permute (AB, C, D and E) in a line and we know there are 4! = 24 ways to arrange 4 units. But A and B could also be stuck together in the order BA, and there will be another 4! arrangements in that case. Therefore, there will be a total of 2 x 4! = 48 arrangements of the 5 people in the line where A and B are always together.

- Example: How many ways are there to sit 6 people around a circular table, where seatings are considered to be the same if they can be obtained from each other by rotating the table?

  Solution: First, place the first person in the north-most chair. This has only one possibility. Then place the other 5 people. There are 5P5 = 5! = 120 ways to do that By the product rule, we get 1x120 =120.

  Alternatively, there are 6P6 = 720 ways to seat the 6 people around the table. For each seating, there are 6 "rotations" of the seating. Thus, the final answer is 720/6 = 120

- Formula for Combinations: In general, to find the number of **combinations** of *r* objects taken from *n* objects, we divide the number of permutations *nPr* by *r!*. The total number of combinations is given by: $^{n}C_{r} = \binom{n}{r} = \frac{^{n}P_{r}}{r!} = \frac{n!}{r!(n-r)!}$

- Example: The manager of a football team has a squad of 16 players. He needs to choose 11 to play in a match. How many possible teams can be chosen?

Solution: This is a combination problem as the order in which the teams are chosen is not important. The number of combinations is ${}^{16}C_{11} = \frac{16!}{11!5!} = 4368$

Lecture 21

Binomial Theorem

- In mathematics we are always looking for generalization of ideas. For example, we know the formula for

  $(x+y)^2 = x^2 + 2xy + y^2$

  $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

  But it is hard to remember formulas for higher powers. We need a mechanism that would help us expand $(x+y)^n$ for any values of n

- Pascal's Triangle:

Expression	Coefficients
$(x+y)^1 = x + y$	1 1
$(x+y)^2 = x^2 + 2xy + y^2$	1 2 1
$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$	1 3 3 1
$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$	1 4 6 4 1
$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$	1 5 10 10 5 1

Each value inside the triangle is obtained by adding the two values above it

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

- Example: Expand $(a + x)^6$

  Solution: Remember the 5th row of the Pascal's triangle: 1 5 10 10 5 1

  Coefficients of the 6th row will be will be: 1, 6, 15, 20, 15, 6 and 1

  So expansion will be $(a + x)^6 = a^6 + 6a^5x + 15a^4x^2 + 20a^3x^3 + 15a^2x^4 + 6ax^5 + x^6$

- We note that the Pascal's triangle is inefficient in finding the coefficient of big expansions such as the coefficient of x^9y^6 in the expansion of $(x + y)^{15}$. What is needed is a formula in terms of n and r for the coefficient of $x^{n-r}y^r$ in the expansion of $(x + y)^n$. We note the following connection between the binomial coefficients and the Pascal's triangle:

$$^{0}C_{0}$$

$$^{1}C_{0} \qquad ^{1}C_{1}$$

$$^{2}C_{0} \qquad ^{2}C_{1} \qquad ^{2}C_{2}$$

$$^{3}C_{0} \qquad ^{3}C_{1} \qquad ^{3}C_{2} \qquad ^{3}C_{3}$$

1

1

1 3

1 4

The n-th row is $^{n}C_{k}$, $k = 0, 1, \ldots, n$

- Binomial Theorem: If n is a natural number, then

$$(x + y)^n = {}^{n}C_0\, x^n + {}^{n}C_1\, x^{n-1}\, y + {}^{n}C_2\, x^{n-2}\, y^2 + {}^{n}C_3\, x^{n-3}\, y^3 + {}^{n}C_4\, x^{n-4}\, y^4 + \ldots + {}^{n}C_n\, y^n = \sum_{k=0}^{n} {}^{n}C_k\, x^{n-k}\, y^k$$

The binomial coefficients are given by $^{n}C_{r} = \dfrac{n!}{r!(n-r)!}$

- Example: Calculate the coefficient of $x^{11}y^4$ in the expansion of $(x+y)^{15}$

Solution: By the binomial theorem the coefficient is given by

$$^{15}C_4 = \frac{15!}{4!11!} = \frac{15.14.13.12}{4.3.2.1} = 1365$$

- Example: Use the Binomial Theorem to calculate 1.01^8

  Solution: $1.01^8 = (1 + 0.01)^8 = 1 + 8(0.01) + 28(0.01)^2 + 56(0.01)^3 + 70(0.01)^4 + 56(0.01)^5 + 28(0.01)^6 + 8(0.01)^7 + (0.01)^8 = 1 + .08 + .0028 + .000056 + .00000070 + .0000000056 + .000000000028 + .00000000000008 + .0000000000000001 = 1.0828567056280801$.

Lecture 22-24

Coordinate Geometry

- Distance Between Two Points: Given any two points $A(x_1, y_1)$ and $B(x_2, y_2)$, the distance between them is given by $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

- Example: Given points $A(2, 3)$ and $B(5, 7)$, find the distance between A and B.
  Solution: $AB = \sqrt{(5-2)^2 + (7-3)^2} = \sqrt{(3)^2 + (4)^2} = 5$

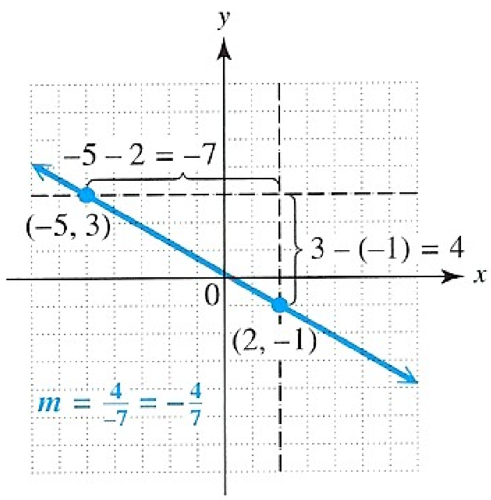
- Mid-Point between two points: Given any two points $A(x_1, y_1)$ and $B(x_2, y_2)$, the mid-point M has coordinates $(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2})$
- Example: Given points $A(4, 3)$ and $B(10, 7)$. Let M be the mid-point of AB. Find the coordinates of M
  Solution: $M = (\frac{4+10}{2}, \frac{3+7}{2}) = (7, 5)$

- Definition: Given any two points $A(x_1, y_1)$ and $B(x_2, y_2)$, the **gradient** or **slope** of the line segment joining A and B is the ratio of the change in y with respect to the change in x. It is denoted by m, and is defined as $m = \frac{y_2 - y_1}{x_2 - x_1}$.
- Properties of Gradient:
  - The bigger the gradient's magnitude is, the steeper the line segment.
  - Negative gradient means line is facing downwards.

- Positive gradient means the line is facing upwards.
- The slope gives the average rate of change in y per unit change in x, where the value of y depends on x.
- Two line segments that are parallel will have the same slope.

- Example: Find the slope of the line through the points (2,-1) and (-5,3)

  Solution: $\text{slope} = m = \dfrac{rise}{run} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{3-(-1)}{(-5)-2} = \dfrac{4}{-7} = -\dfrac{4}{7}$



- Equation of a Line: The equation of a line or a curve is a rule for determining whether or not the point with coordinates (x, y) lies on the line or curve. The equation of a line through a fixed point (x_1, y_1) with gradient m is given by

  $$\frac{y - y_1}{x - x_1} = m$$
  $$\Leftrightarrow y - y_1 = m(x - x_1)$$
  $$\Leftrightarrow y = mx + y_1 - mx_1$$
  $$\Leftrightarrow y = mx + c$$

  where $c = y_1 - mx_1$

- The last form of the line is called the **slope-intercept** form of a straight line where m is the slope and c is the y-intercept.

- Example: Find the slope of the line given $3x - 4y = 12$

  Solution: Writing the equation in the slope-intercept form, we get:

  $$3x - 4y = 12$$
  $$-4y = -3x + 12$$
  $$y = \frac{3}{4}x - 3$$

  Therefore, the slope is ¾.

- Two Lines in a Plane: In a plane, two lines either intersect or are parallel. If the gradients are the same, the lines are parallel and do not intersect. If the gradients are not the same, the lines intersect.

- Example: Given the equations of the lines $2x - y = 4$ and $3x + 2y = -1$. Do these lines intersect? If they do, find the point of intersection.

  Solution: Since these two lines have different gradients, they must intersect. To find the point of intersection we need (x, y) which lie on both the lines, i.e. which satisfy the two equations simultaneously. We need to solve the equations simultaneously. Solving the equations simultaneously we get $x = 1$ and $y = -2$.

- Perpendicular Lines: If a line has gradient m, then the gradient of a line perpendicular to this line is $-1/m$. Two lines with gradients m_1 and m_2 are perpendicular if $m_1.m_2 = -1$, or $m_1=-1/m_2$

- Example: Are the lines $3x + 5y = 6$ and $5x - 3y = 2$ parallel, perpendicular, or neither?

  Solution: For Line 1: $x+5y=6 \Leftrightarrow 5y=-3x+6 \Leftrightarrow y=-\dfrac{3}{5}x+\dfrac{6}{5}$

  So, its slope is -3/5.

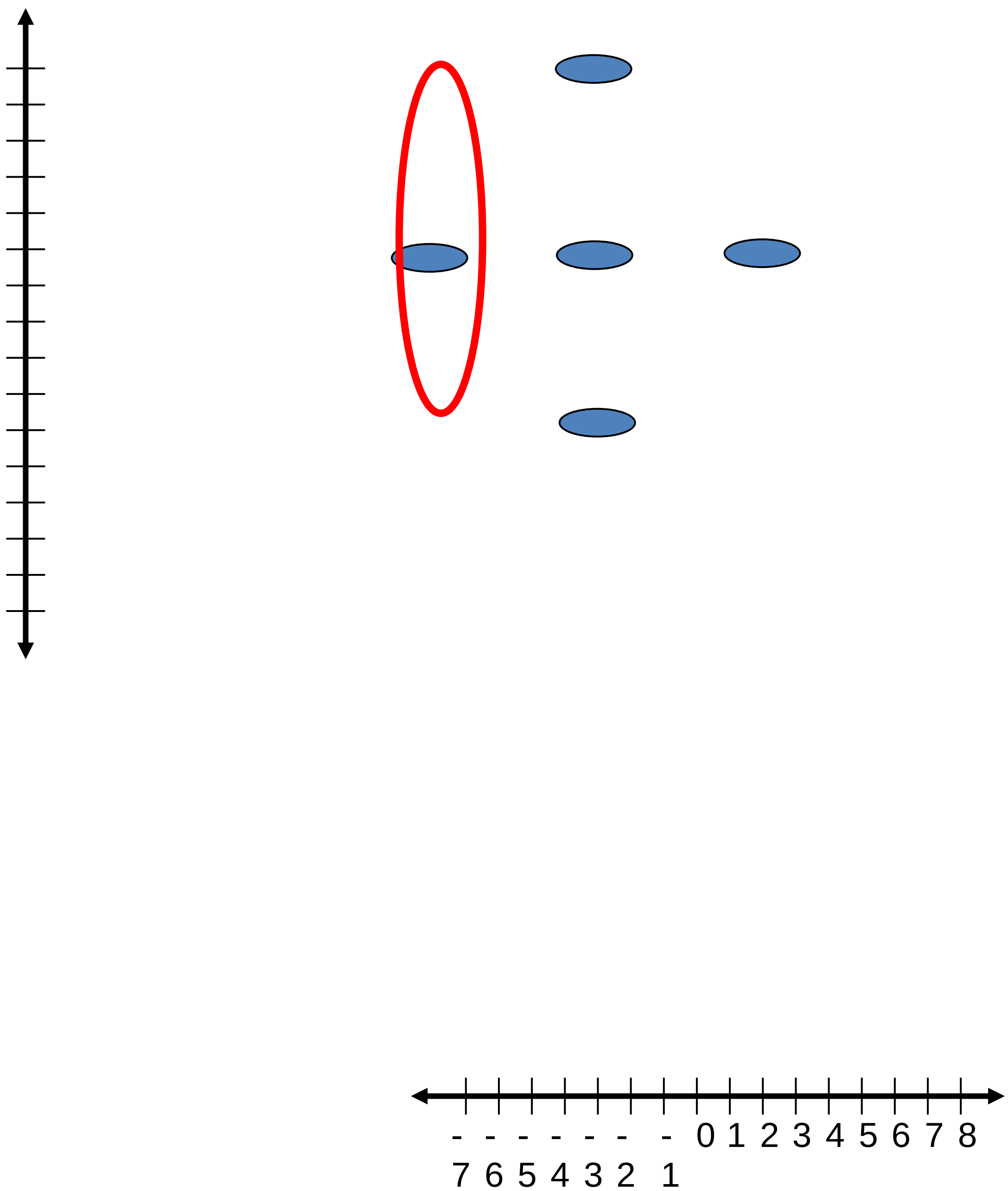
  For Line 2: $5x - 3y = 2 \Leftrightarrow -3y=-5x+2 \Leftrightarrow y=\dfrac{5}{3}x-\dfrac{2}{3}$

  So, its slope is 5/3, which is the negative reciprocal of the slope of the first line. Therefore, the two lines are perpendicular.

- Equation of a Circle: The equation of a circle with center (h, k) and radius r in standard form is: $(x-h)^2 + (y-k)^2 = r^2$

- Example: Identify the center and radius of the given circle and sketch the graph: $(x+4)^2+(y-3)^2=25$

  Solution: Comparing the given equation with standard form, we see that its center is (-4, 3) and radius is 5. The graph is as follows:



- Example: Find the center and radius of the circle with equation $x^2 + y^2 + 6x - 4y = 23$

  Solution: We transform the equation into the form $(x - h)^2 + (y - k)^2 = r^2$ by completing the square relative to x and relative to y. Then from this standard form we can determine the center and radius.

  We want to write $x^2 + y^2 + 6x - 4y = 23$ as
  $(x^2 + 6x \quad) + (y^2 - 4y \quad) = 23$
  $(x^2 + 6x + \mathbf{9}) + (y^2 - 4y + \mathbf{4}) = 23 + \mathbf{9} + \mathbf{4}$
  $(x+3)^2 + (y - 2)^2 = 36$

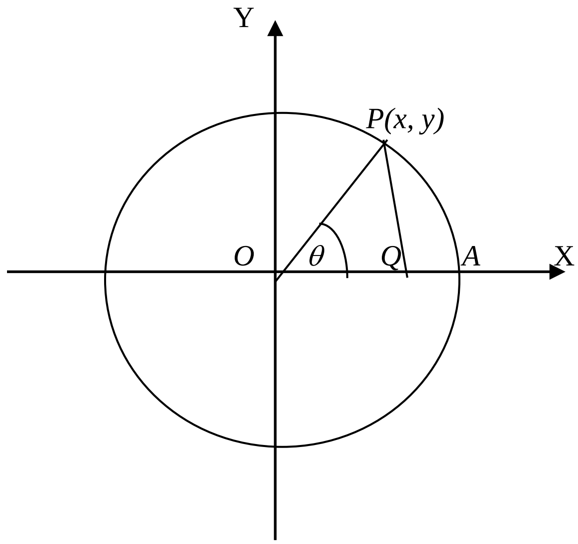
$(x-(-3))^2 + (y-2)^2 = 6^2$

The center is *(-3, 2)* and the radius is *6*

Lecture 24-30

Trigonometry

- Basic Functions:
  - Cosine Function: Consider a circle of radius 1.

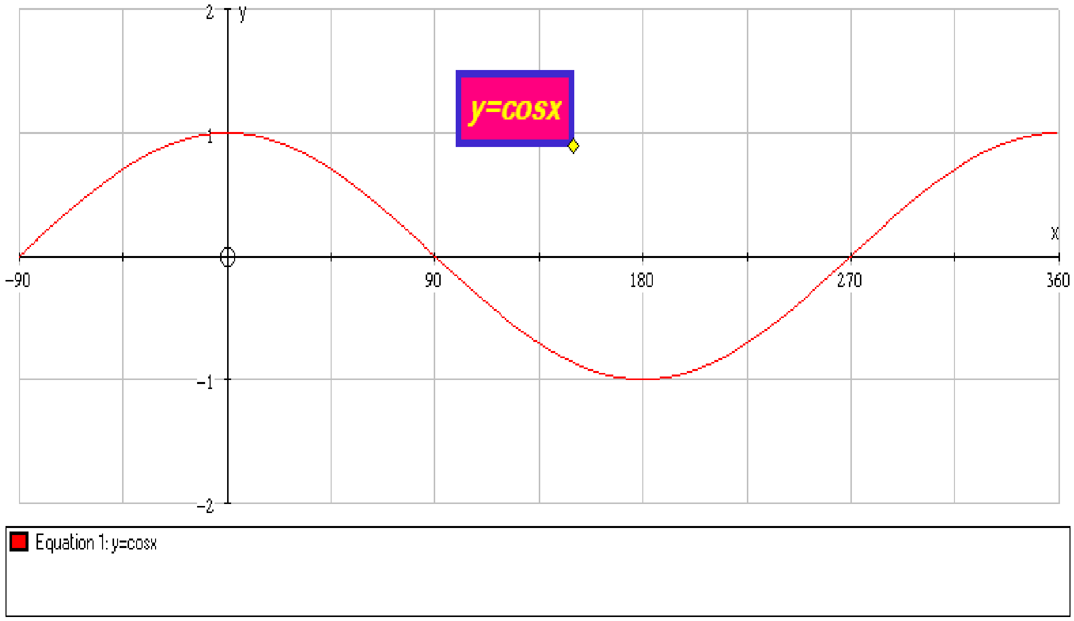


Let $P(x, y)$ be any point making an angle θ with the horizontal x-axis.
Then $Cos\theta = OQ/OP = x/1 = x$
i.e. the value of $Cos\theta$ is the x-coordinate of P as P travels along the circumference of the circle, starting from the point A.

- Properties of the Cosine Function:
  - The cosine function has "period" 360^o as it repeats itself after each revolution of 360^o
  - $\text{Cos}(-\theta) = \text{Cos}\theta$ as the x-coordinate of P doesn't change when we reflect across the x-axis.
  - $\text{Cos}(180 - \theta) = -\text{Cos}\theta$ as the x-coordinate changes signs when reflected across the y-axis.
  - $\text{Cos}(\theta - 180) = -\text{Cos}\theta$ as the x-coordinate changes signs when reflected across the origin.
  - $\text{Cos}\theta$ is positive in the first and the fourth quadrant (as the x coordinate of P is positive), and negative in the second and the third quadrant as the x-axis is negative there.
  - The range of the cosine function is between -1 and 1. The maximum value of 1 is taken when $\theta = 0^o, \pm 360^o, \pm 720^o, \ldots$, and the minimum value of -1 is at $\theta = \pm 180^o, \pm 540^o, \ldots$,

- Definition: The functions with the property that they keep repeating themselves are called **periodic functions**. The smallest interval for which the function repeats itself is called its **period**.

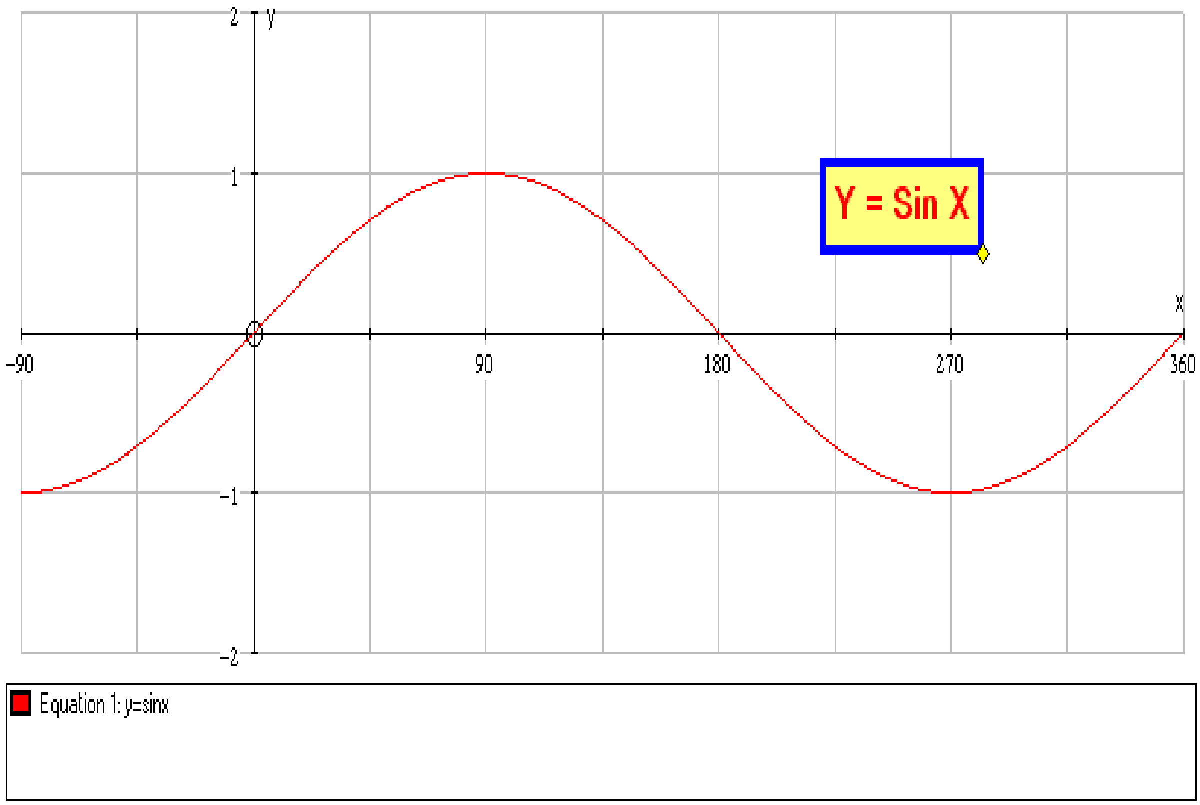
- Graph of the Cosine Function:



- Sine Function: Using the same diagram as before, we consider a circle of radius 1. Let P(x, y) be any point making an angle θ with the horizontal x-axis. Then Sinθ = PQ/OP = y/1 = y
  i.e. the value of Sinθ is the y-coordinate of P as P travels along the circumference of the circle, starting from the point A.

- Properties of the Sine Function:
  - Sin(-θ) = -Sinθ as the y-coordinate of P changes sign when we reflect across the x-axis.
  - Sinθ is positive in the first and the second quadrant (as the y coordinate of P is positive), and negative in the third and the fourth quadrant as the y coordinate is negative there.
  - Sin(180 - θ) = Sin θ because as we reflect across the y-axis the y-coordinate doesn't change.
  - Sin(θ - 180) = - Sinθ as the y-coordinate changes signs when reflected across the origin.
  - Like the cosine function, the sine function is also periodic, with period 360 degrees, and range between -1 and 1.

- Graph of the Sine Function:

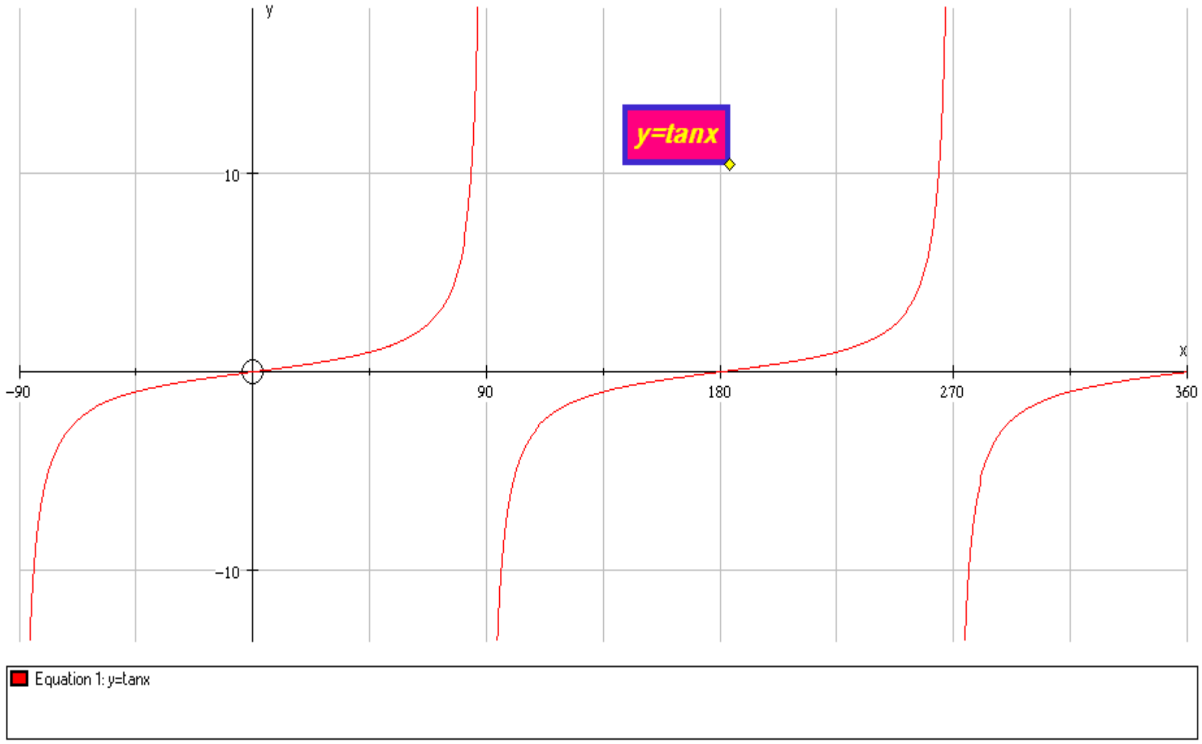


- Tangent Function: Using the same diagram as before, we consider a circle of radius 1. Let *P(x, y)* be any point making an angle θ with the horizontal *x*-axis.

  Then $Tan\theta = PQ/OQ = y/x = sin\theta / cos\theta$

- Properties of the Tangent Function:
  - Tan(-θ) = -Tanθ as the y-coordinate of P changes sign when we reflect across the x-axis but the x-coordinate doesn't change sign.
  - Tanθ is positive in the first and the third quadrant (as the x and y coordinates of P have the same signs in these quadrants), and negative in the second and the fourth quadrant as the x and y coordinates have opposite signs in these quadrants.
  - Tan(180 -θ) = -Tanθ as the x-coordinate of P changes sign when we reflect across the y-axis but the y-coordinate doesn't change sign
  - The domain of tanθ does **not** include the angles for which x is 0, namely, for θ = ±90, ±270, …
  - Like the cosine and sine functions, the tangent function is also periodic, but its period is 180. i.e. tan(θ + 180) = tanθ and tan(θ - 180) = tanθ

  - Graph of the Tangent Function:



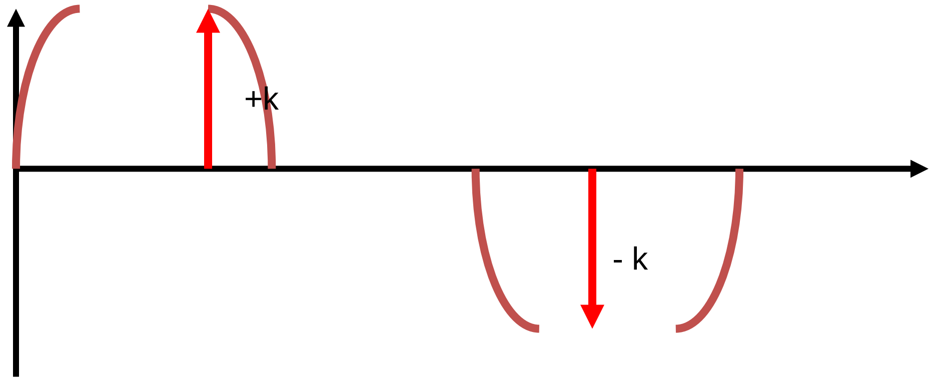
- Basic Properties of Trigonometric Functions:

Function	Periodic Properties	Odd/Even Property	Translation Properties	
Cosine	Cos $(\theta \pm 360)$ = Cosθ	Cos$(-\theta)$ = Cosθ	Cos $(\theta - 180)$ = -Cosθ Cos $(180 - \theta)$ = -Cosθ	
Sine	Sin $(\theta \pm 360)$ = Sinθ	Sin$(-\theta)$ = -Sinθ	Sin $(\theta - 180)$ = -Sinθ Sin $(180 - \theta)$ = Sinθ	
Tangent	Tan $(\theta \pm 180)$ = Tanθ	Tan$(-\theta)$ = -Tanθ	Tan $(180 - \theta)$ = -Tanθ	

- Definition: The **Amplitude** of a function is the height from the mean (or the rest) value of the function to its maximum or minimum value.

    - The amplitude of the function $f(x) = ASinBx$ is $|A|$ and the period is $2\pi/|B|$
    - The amplitude of the function $g(x) = ACosBx$ is $|A|$ and the period is $2\pi/|B|$

- Changing Trigonometric Graphs:
  - *y= k Cosx & y = k Sinx:* The amplitude of the function is "*k*" .



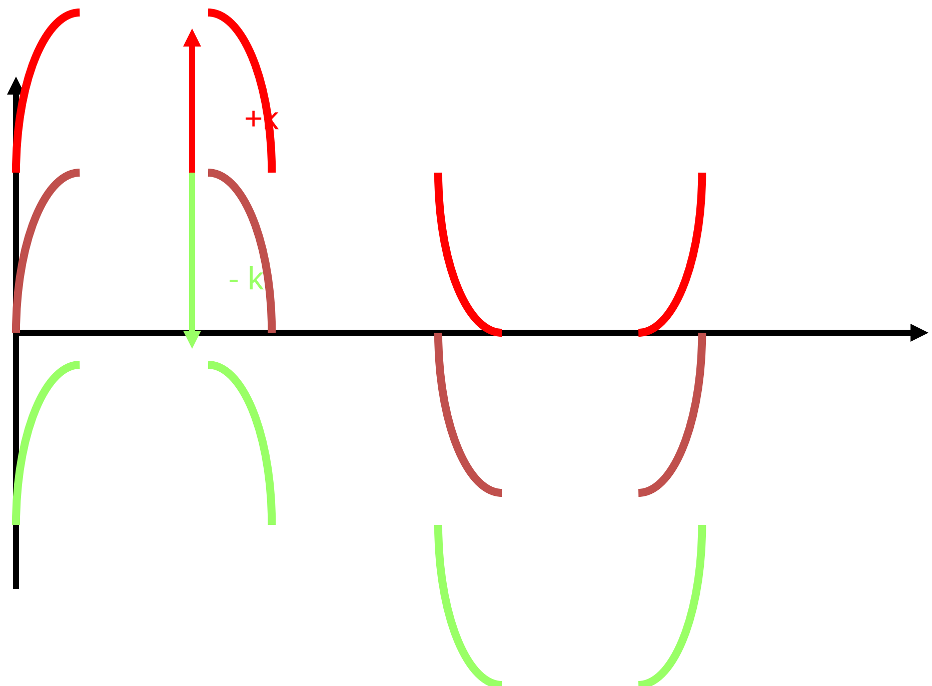
  - *y = Coskx & y = Sinkx:* The period of the function is "*360 ÷ k*".

3
6
0

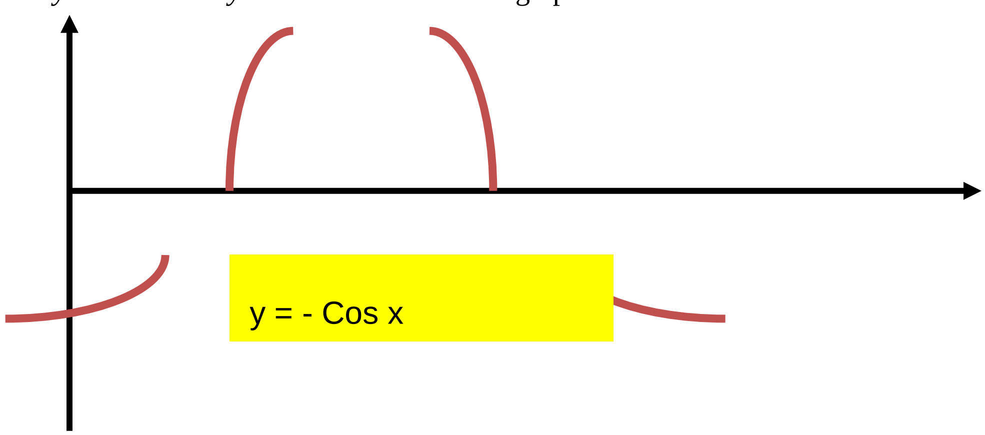
÷

K

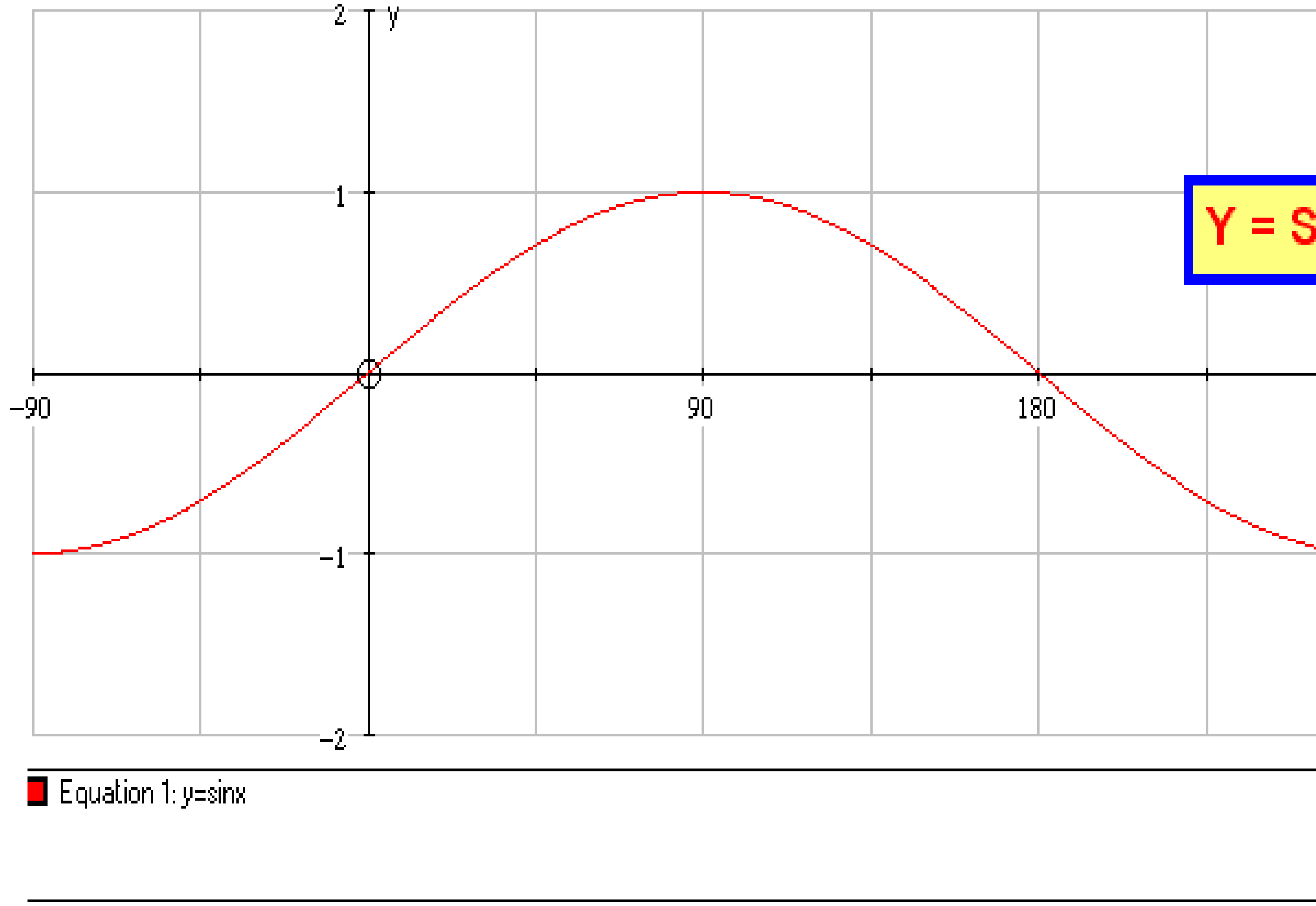
  - *y= Cos x + k & y = Sin x + k:* Translates the graph + *k* or – *k* parallel to the *y*-axis.



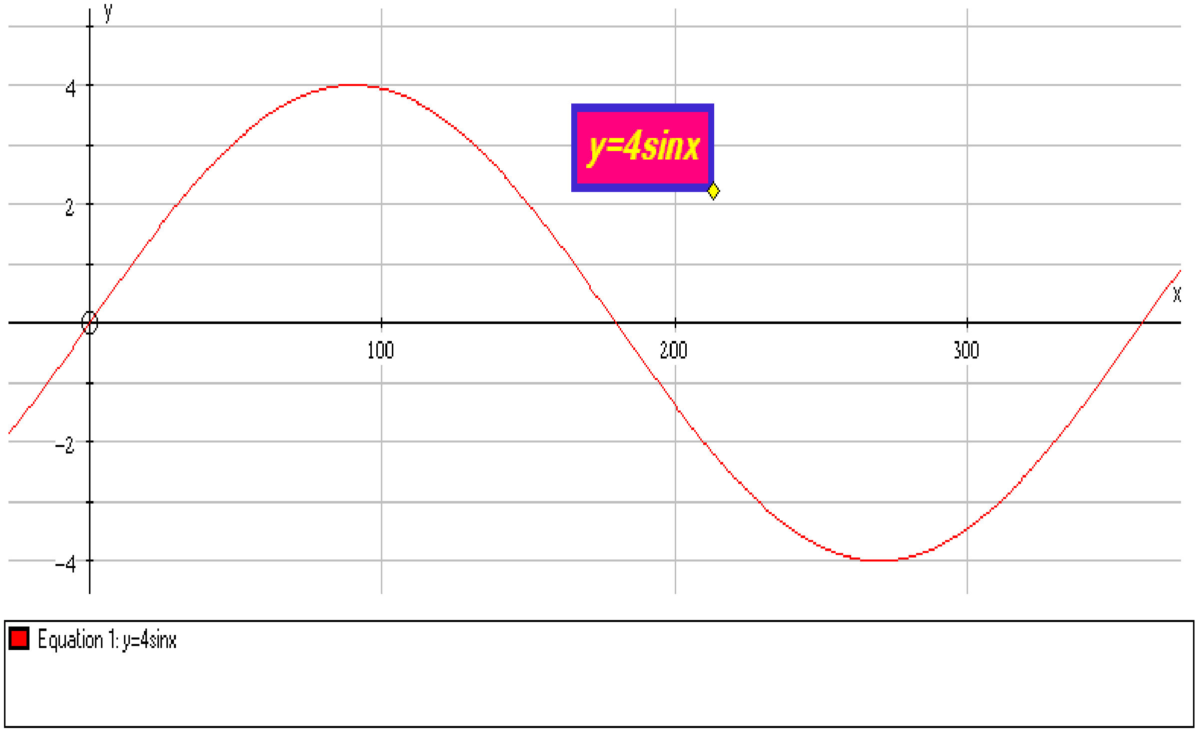
- $y = -\,Cos\ x$ & $y = -\,Sin\ x$: Reflects the graph in the *x*-axis.



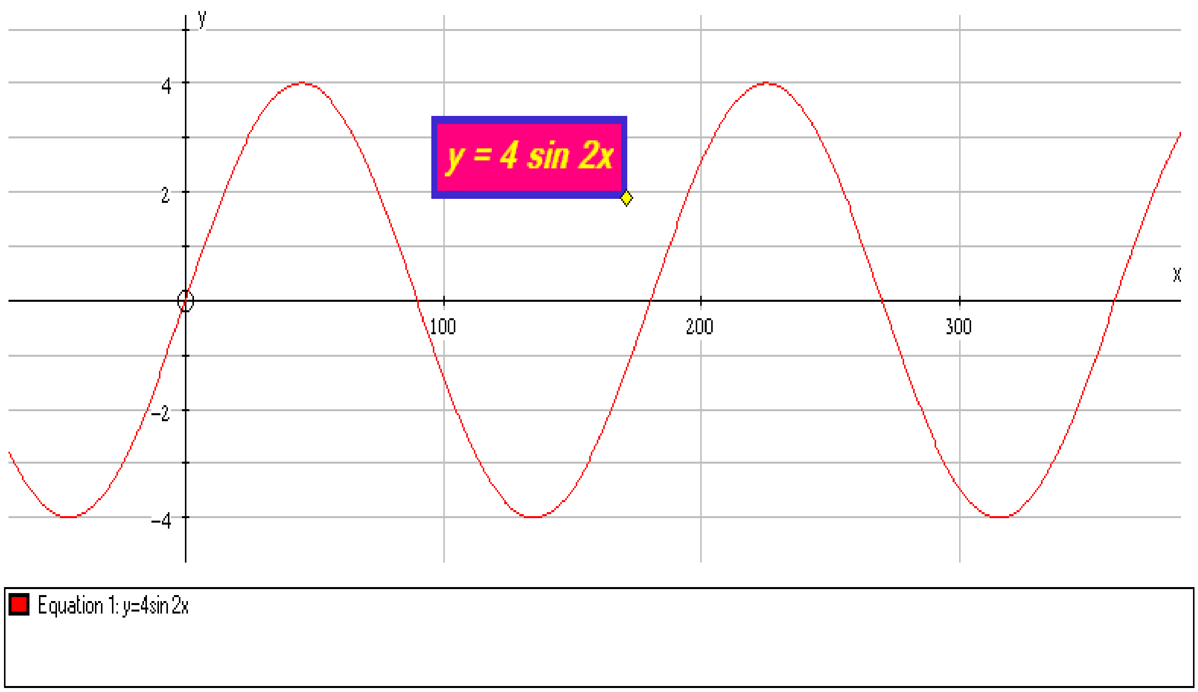
- Combining the Effects: Draw the graph of $y = 4sin2x + 3$.
    - Step 1: Draw the graph of $y = sin\ x$



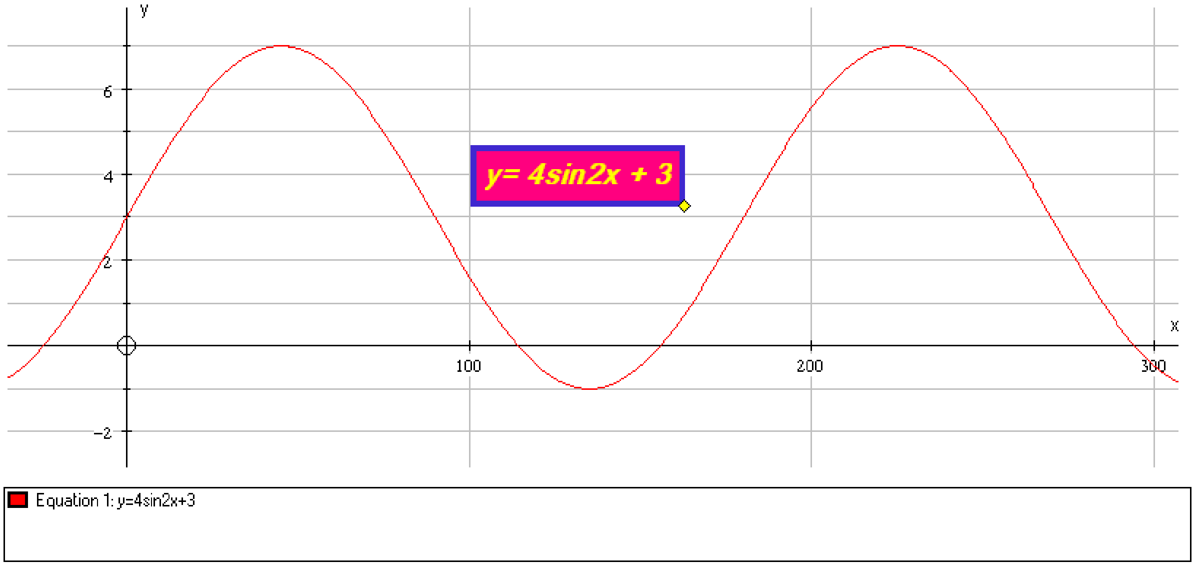
- Step 2: Draw the graph of $y = 4 \sin x$



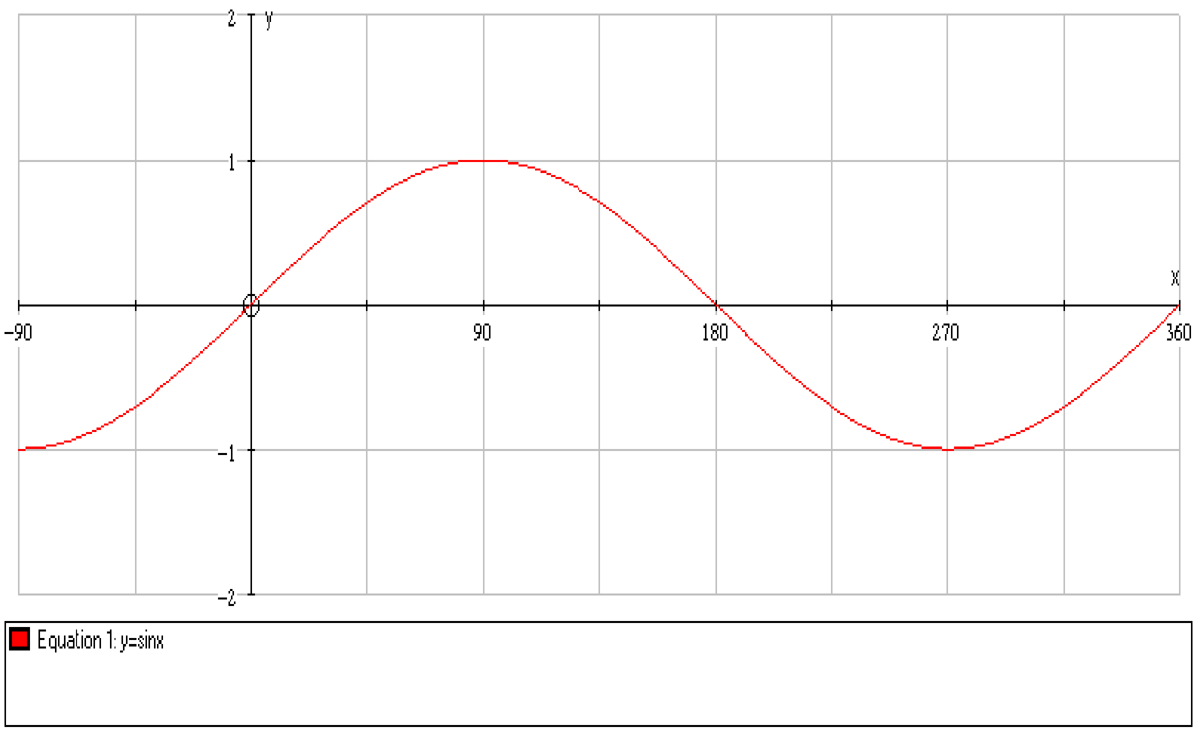
- Step 3: Draw the graph of $y = 4 \sin 2x$



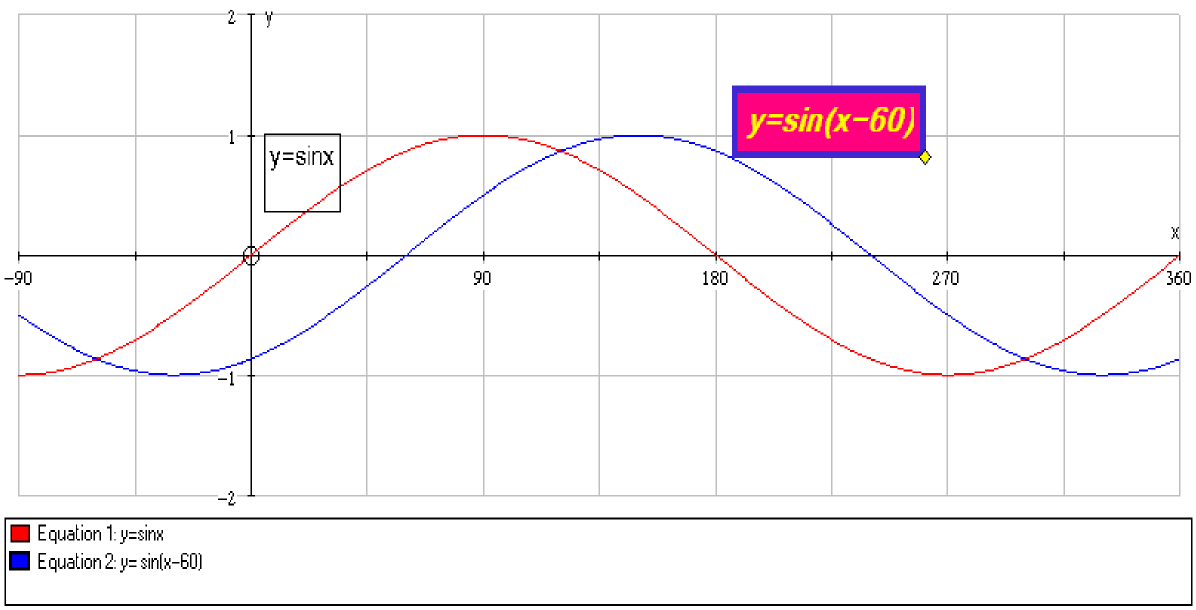
- Step 4: Draw the graph of $y = 4 \sin 2x + 3$



- Phase Shift:
  - Shown below is the graph of $y = \sin x^o$



  - Now compare it with the graph of $y= \sin(x - 60^o)$



The graph is translated 60^o to the right parallel to the x axis.

- From the previous example we can now see that the equation $y = cos\ (x - k)$ and $y = sin\ (x - k)$ translate the graphs k^o to the right parallel to the *x*-axis and the equations $y = cos\ (x + k)$ and $y = sin\ (x + k)$ translate the graph k^o to the left parallel to the *x*-axis.

- Note that the functions Sine, Cosine and Tangent are not one-to-one. So, they don't have inverses unless we restrict the domains of the definitions.

- Definition: The **inverse sine function** is defined by $y = \arcsin x$ if and only if $\sin y = x$. The domain of $y = \arcsin x$ is $[–1, 1]$. The range of $y = \arcsin x$ is $[–\pi/ , \pi/2]$.

- Example: $\arcsin \frac{1}{2} = \frac{\pi}{6}$ since $\frac{\pi}{6}$ is the angle whose sine is $\frac{1}{2}$.

- Example: $\sin^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{3}$ since $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$

- Definition: The **inverse cosine function** is defined by $y = \arccos$ if and only if $\cos y = x$. The domain of $y = \arccos x$ is $[–1, 1]$. The range of $y = \arccos x$ is $[0, \pi]$.

- Definition: The **inverse tangent function** is defined by $y = \arctan x$ if and only if $\tan y = x$. The domain of $y = \arctan x$ is $(-\infty, \infty)$. The range of $y = \arctan x$ is $[–\pi/2, \pi/2]$.

- Solving Trigonometric Equations
  - To solve $\cos\theta = k$
    - Find $\cos^{-1}(k) = \theta$
      - Use symmetry property to get $\cos(-\theta) = \cos(\theta)$ to get $-\theta$ as a solution.
      - Use the periodic property [$\cos(\theta \pm 360) = \cos\theta$] to find all the solutions in the required interval.
    - Example: Given that $\cos(70.52) = 1/3$, solve $\cos\theta = 1/3$, giving all the roots in $0 \le \theta \le 360$
      - Find $\cos^{-1}(1/3) = 70.52$
      - Use symmetry property to get $\cos(-70.52) = \cos(70.52)$ to get -70.52 as a solution.
      - Note that -70.52 is not in the required interval. Now use the periodic property $\cos(-70.52) = \cos(-70.52 + 360) = \cos(289.48)$. So another solution is 289.48 degrees
      - All the solutions are 70.52° and 289.48°
    - To solve $Sin\theta = k$
      - Find $\sin^{-1}(k) = \theta$
      - Use symmetry property to get $\sin(180 - \theta) = \sin(\theta)$ to get another solution.
      - Use the periodic property [$\sin(\theta \pm 360) = \cos\theta$] to find all the solutions in the required interval.
    - Example: Given that $\sin(44.42^{\circ}) = 0.7$, solve $\sin\theta = -0.7$, giving all the roots in $-180 \le \theta \le 180$
      - Find $\sin^{-1}(-0.7) = -44.42$
      - Use symmetry property to get $\sin(180 - (-44.42)) = \sin(-44.42)$ to get 224.42 as a solution.
      - Note that 224.42 is not in the required interval. Now use the periodic property $\sin(224.42) = \sin(224.42 - 360)$. So another solution is -135.58 degrees
      - All the solutions are -44.42° and -135.58°
    - To solve $Tan\ \theta = k$
      - Find $Tan^{-1}(k) = \theta$
      - Use the periodic property $\tan(180 \pm \theta) = \tan(\theta)$ to get other solutions in the required interval.
    - Example: Given that $\tan(63.43^{\circ}) = 2$, Solve $\tan\theta = -2$, giving all the roots *in* $0 \le \theta \le 360$
      - Find $\tan^{-1}(-2) = -63.43^{\circ}$,
      - Note that -63.43° is not in our range. Now use the periodic property $\tan(-63.43^{\circ}) = \tan(-63.43^{\circ} - 180^{\circ})$ to get 116.56. Now use the periodic property $\tan(116.56^{\circ}) = \tan(116.56^{\circ} + 180^{\circ})$ to get 296.56°.
      - The solutions are is 116.6° and 296.6°

- Definition: Two functions *f* and *g* are said to be **identically equal** if *f(x) = g(x)* for **every value** of x for which both functions are defined. Such an equation is referred to as an **identity**. An equation that is not an identity is called a **conditional equation.**

- Basic Trigonometric Identities:

  - Reciprocal Identities:

$$\csc\theta = \frac{1}{\sin\theta} \quad \sec\theta = \frac{1}{\cos\theta} \quad \cot\theta = \frac{1}{\tan\theta}$$

  - Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

  - Periodic Properties

$$\sin(\theta + 2\pi) = \sin\theta \qquad \csc(\theta + 2\pi) = \csc\theta$$
$$\cos(\theta + 2\pi) = \cos\theta \qquad \sec(\theta + 2\pi) = \sec\theta$$
$$\tan(\theta + \pi) = \tan\theta \qquad \cot(\theta + \pi) = \cot\theta$$

  - Even-Odd Properties

$$\sin(-\theta) = -\sin\theta \qquad \csc(-\theta) = -\csc\theta$$
$$\cos(-\theta) = \cos\theta \qquad \sec(-\theta) = \sec\theta$$
$$\tan(-\theta) = -\tan\theta \qquad \cot(-\theta) = -\cot\theta$$

  - Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

- Example: Verify that the following equation is an identity:

$$\frac{\tan t - \cot t}{\sin t \cos t} = \sec^2 t - \csc^2 t$$

Solution: LHS = $\dfrac{\tan t - \cot t}{\sin t \cos t} = \dfrac{\tan t}{\sin t \cos t} - \dfrac{\cot t}{\sin t \cos t}$

$$= \tan t \cdot \frac{1}{\sin t \cos t} - \cot t \cdot \frac{1}{\sin t \cos t}$$

$$= \frac{\sin t}{\cos t} \cdot \frac{1}{\sin t \cos t} - \frac{\cos t}{\sin t} \cdot \frac{1}{\sin t \cos t}$$

$$= \frac{1}{\cos^2 t} - \frac{1}{\sin^2 t}$$

$$= \sec^2 t - \csc^2 t$$

= RHS

- Sum and Difference Formulas:
  - cos(A-B) = cosAcosB + sinAsinB
    cos(A+B) = cosAcosB - sinAsinB
  - sin(A+B) = sinAcosB + cosAsinB
    sin(A-B) = sinAcosB – cosAsinB
  - $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

    $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

- Double Angle Formulas:
  - $\sin 2\text{A} = \sin(\text{A}+\text{A}) = \sin\text{A}\cos\text{A} + \sin\text{A}\cos\text{A} = 2\sin\text{A}\cos\text{A}$
  - $\cos 2\text{A} = \cos(\text{A}+\text{A}) = \cos\text{A}\cos\text{A} - \sin\text{A}\sin\text{A} = \cos^2\text{A} - \sin^2\text{A}$
  - $\tan 2\text{A} = \dfrac{\tan\text{A} + \tan\text{A}}{1 - \tan\text{A}\tan\text{A}} = \dfrac{2\tan\text{A}}{1 - \tan^2\text{A}}$

- Example: Find the exact value of Cos 165^{o}.

  Solution: $\cos(165°) = \cos(210° - 45°) = \cos 210° \cos 45° + \sin 210° \sin 45°$

  $$= \frac{-\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} + -\frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{-\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{-(\sqrt{6} - \sqrt{2})}{4}$$

Using the trigonometric identities find the value of $\cos(75°)$

Lecture 31- Review Sequence and Series

Lecture 32- Review Permutation and Combination, Binomial Theorem

Lecture 33- Review Binomial Theorem, Coordinate Geometry

Lecture 34- Coordinate Geometry, Trigonometry

Lecture 35- Application of Mathematics in different Fields

Lecture 36

Statistics

- Definition: Methods of collection, organization and analysis of numerical information are collectively called **statistics.** Pieces of numerical and non-numerical information are called **data.** In order to collect data, you need to observe or measure some property; this property is called a **variable**.

- Definition: A variable is **qualitative** if it is not possible for it to take a numerical value. A variable is **quantitative** if it can take a numerical value. A quantitative variable which can take any value in a given range is called a **continuous** variable. A quantitative variable which has clear steps between its possible values is called a **discrete** variable.

- Types of Statistics: **Descriptive Statistics** comprises those methods concerned with collection and describing a set of data so as to yield meaningful information.

    - Example: Summarized large amounts of data collected from the pool games of 2011 cricket world cup to provide immediate meaningful information concerning the performance of each team.

Group A

Teams	Mat	Won	Lost	Tied	N/R	Pts	Net RR	For	Against
Pakistan	6	5	1	0	0	10	+0.758	1312/275.1	1155/288.0
Sri Lanka	6	4	1	0	1	9	+2.582	1336/218.4	882/250.0
Australia	6	4	1	0	1	9	+1.123	1181/218.5	1030/241.0
New Zealand	6	4	2	0	0	8	+1.135	1257/241.3	1156/284.0
Zimbabwe	6	2	4	0	0	4	+0.030	1288/288.0	1189/267.4
Canada	6	1	5	0	0	2	-1.987	1054/295.3	1582/284.5
Kenya	6	0	6	0	0	0	-3.042	932/300.0	1366/222.1

Group B

Teams	Mat	Won	Lost	Tied	N/R	Pts	Net RR	For	Against
South Africa	6	5	1	0	0	10	+2.026	1595/292.3	1028/300.0
India	6	4	1	1	0	9	+0.900	1673/282.3	1505/299.4
England	6	3	2	1	0	7	+0.072	1600/298.4	1576/298.1
West Indies	6	3	3	0	0	6	+1.066	1299/262.2	1138/292.5
Bangladesh	6	3	3	0	0	6	-1.361	1017/290.2	1276/262.2
Ireland	6	2	4	0	0	4	-0.696	1393/296.5	1595/296.0
Netherlands	6	0	6	0	0	0	-2.045	1182/300.0	1641/274.1

- **Inferential Statistics** comprises those methods concerned with analysis of a subset of data leading to predictions or inferences about the entire set of data.

    - Example: Suppose we collected data for 30 years regarding the average rainfall in the month of July in Lahore, and the amount came to be 3.3 centimeters. We make the inference that next year in the month of July we can expect between 3.2 and 3.4 centimeters of rain.

Where can Statistics Contribute?

- Business and Industry
    - Manufacturing
        - Improve product quality
        - Increase efficiency of processes
    - Marketing
        - Conduct sample surveys
        - Determine product viability
        - Estimate advertisement effectiveness
    - Engineering
        - Make consistent product
        - Predict product life
    - Banking
        - Estimate the risk of a company defaulting on loan
        - Determine effective asset allocation for portfolios

- Health and Medicine
    - Epidemiology
        - Calculate cancer incidence rates
        - Monitor disease outbreaks
        - Study risk factors for various diseases
    - Public Health
        - Design community efforts
        - Education programs
    - Pharmacology
        - Drug discovery, development, and approval
        - Ensure validity of results in clinical trials
    - Genetics
        - Identify potential indicators for specific diseases or traits
        - Test gene modification for treatment of diseases

- Natural Resources
    - Agriculture
        - Evaluate differences in crop management
        - Evaluate the best combinations of fertilizers, pesticides and densities of planting.
    - Ecology
        - Study changes in local and global climate
        - Develop strategies to improve the environment
        - Study the impact of new industrial plants on surrounding ecology
    - Geography
        - Evaluate the amount of rainfall one can expect for a given area based on longitude, latitude and distance from the sea.

- Social and Natural Sciences
    - Physics
        - Determine when an increase in the density of cosmic rays signals the presence of a supernova
        - Conducting tests to determine existence of new particles.

- Chemistry
  - Predict shape of large molecules
  - Analysis of mass spectrometry data
- Biology
  - Seek to better understand why insects cluster
  - Identify genes related to a particular disease
- Sociology
  - Estimate the chances of a major war in the next five years
  - Study the increase in rates of marriage failure
  - Determine the characteristic of prisoners to study risk of them repeating criminal behavior.
- Psychology
  - Study the effects of narcotics on schizophrenia
  - Determine if the existence of extrasensory perception can be demonstrated
  - Evaluate the relationship between shyness and loneliness
- Anthropology
  - Determine the age of an archaeological site
  - Analyze the percentage difference in body fat between urban and rural dwellers in Pakistan
- Zoology
  - Evaluate the differences in behavior of caged animals when they are outdoors and when they are indoors
  - Determine what techniques are more effective in counting a given species of bird.

- Education
  - Develop effective teaching strategies
  - Research appropriate and informative evaluation (testing) instruments
  - Identify risk factors for bullying, dropping out, failing
  - Identify factors contributing to a decrease/increase in student achievement
  - Study the proportion of graduates of various programs subsequently employed in their field of study

- Government Media and Law
  - Government Agencies
    - Design and implement effective sampling strategies
    - Estimate the unemployment rate
    - Track and report changes in the economy
    - Regulations on stock trading, drug approval, pollution
  - Journalism
    - Effective communication of statistical ideas to a broad audience
    - Participation in education efforts
  - Expert Witness
    - Testify in court cases involving DNA evidence, salary discrepancies, discrimination, and disease clusters

- Definitions: **A population** is defined as the set of all possible members of a stated group. A cross-section of the returns of all of the stocks traded on the New York Stock Exchange (NYSE) is an example of a population.

- Definition**: A sample** is defined as a subset of the population of interest. Once a population has been defined, a sample can be drawn from the population, and the sample's characteristics can be used to describe the population as a whole.

- Definition: A measure used to describe a characteristic of a population is referred to as a **parameter**.

- Definition: It is frequently too costly or time consuming to obtain measurements for every member of a population, if it is even possible. In this case, a sample may be used. In the same manner that a parameter may be used to describe a characteristic of a population, a **sample statistic** is used to measure a characteristic of a sample.

- Measurements Scales:

    - **Nominal scale:** Observations are classified or counted with *no particular order*. It consists of assigning items to groups or categories. *No quantitative information* is conveyed and *no ordering* of the items is implied. Nominal scales are therefore *qualitative* rather than quantitative.
        - Religious preference, race, and gender are all examples of nominal scales.

    - **Ordinal scale:** All observations are placed into separate categories and the *categories are placed in order with respect to some characteristic*. Differences between values makes no sense.
        - Political parties on left to right spectrum given labels 0, 1, 2; restaurant ratings, etc, are examples of ordinal scales.

    - **Interval scale:** This scale provides *ranking* and assurance that differences between scale values are equal. *Difference makes sense*, but ratio doesn't; and there is *no natural zero*.
        - temperature (C,F) and dates are examples of interval scale

    - **Ratio scale:** These represent the strongest level of measurement. In addition to providing ranking and equal differences between scale values, ratio scales have a *true zero point as the origin*.
        - Height, weight, age and length are all examples of ratio scale.

Lecture 37

Representation of Data

- Definition: A **frequency distribution** is a tabular presentation of statistical data that aids the analysis of large data sets. Frequency distributions summarize statistical data by assigning it to specified groups, or intervals.

  Three steps are required to construct a frequency distribution:
    - Define the intervals.
    - Tally the observations.
    - Count the observations.

- Definition: **Relative frequency** is calculated by dividing the frequency of each interval by the total number of observations. Simply, relative frequency is the percentage of total observations falling within each interval.

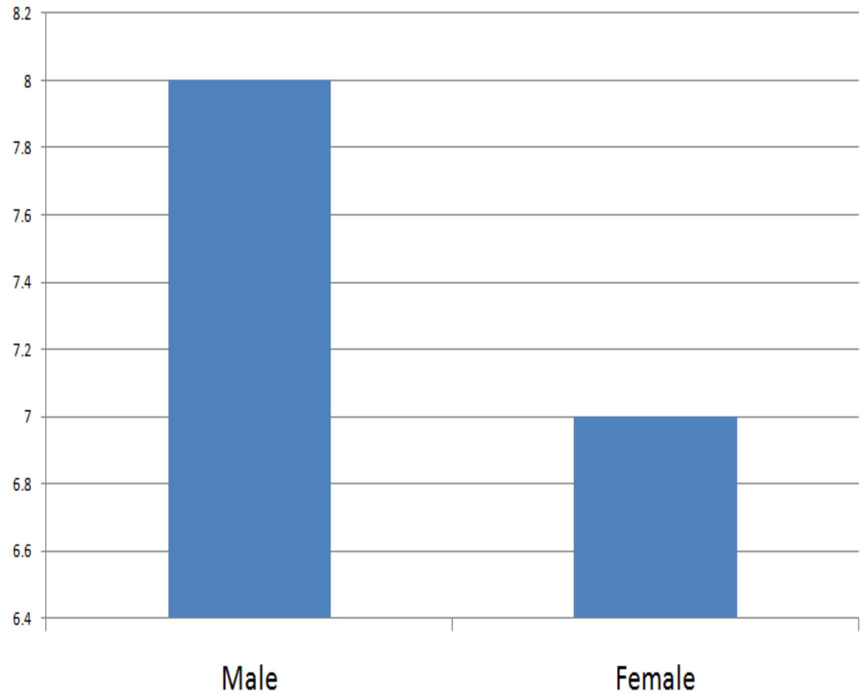
- Definition**: Cumulative Frequency** is calculated by summing the frequencies starting at the lowest interval and progressing through the highest. Cumulative frequency for any given interval is the sum of the frequencies up to and including the given interval.

- Example:

Height Range	Frequency	Cumulative Frequency	Relative Frequency	Cumulative Relative Frequency
4.5 – 5.0 ft	25	25	0.25	0.25
5.0 – 5.5 ft	35	60	0.35	0.60
5.5 – 6.0 ft	29	89	0.29	0.89
6.0 – 6.5 ft	11	100	0.11	1.00

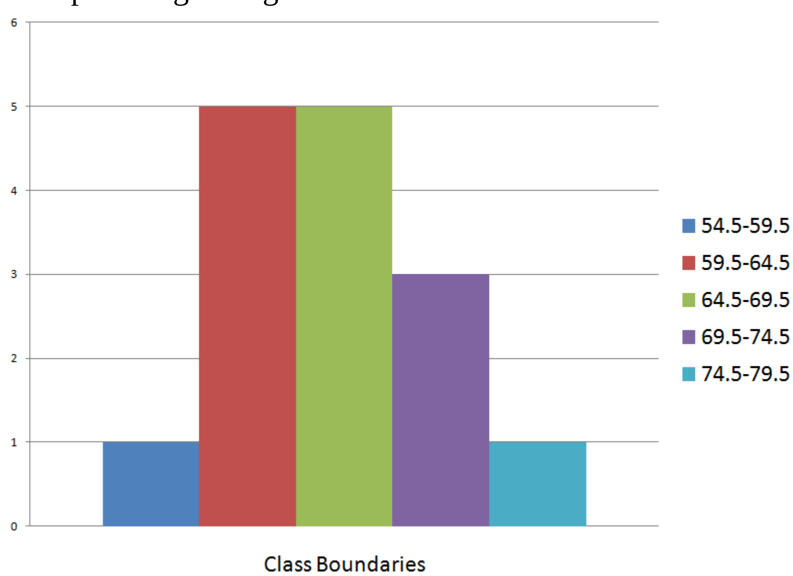
- Definition: A **Bar chart** graphically represents the data sets by representing the frequencies as heights of bars.

- Example: Genderwise groupings of students.



- Definition: A Bar chart which represents continuous data is called a **histogram** if
  - The bars have no spaces between them (though there may be bars with zero height, which could look like spaces).
  - The area of each bar is proportional to the frequency.
  - If all the bars have the same width, then the height is proportional to the frequency.

- Example: Height ranges of students in centimeters.

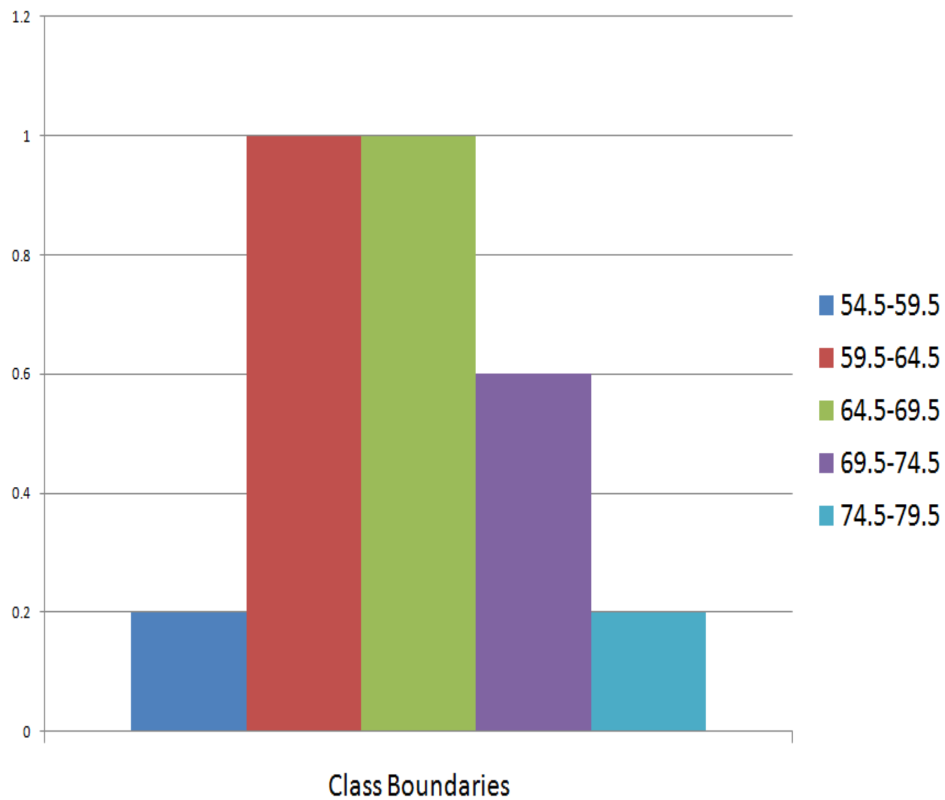


- Definition: **Frequency Density** is defined as the ratio between the frequency of a class and the class width. i.e. Frequency Density = Frequency/Class Width

- Example:

Height Range	Class Width	Frequency	Frequency Density
54.5 – 59.5	5	1	0.2
59.5 – 64.5	5	5	1.0
64.5 – 69.5	5	5	1.0
69.5 – 74.5	5	3	0.6
74.5 – 79.5	5	1	0.2

- Histogram: Frequency Density of height ranges of students

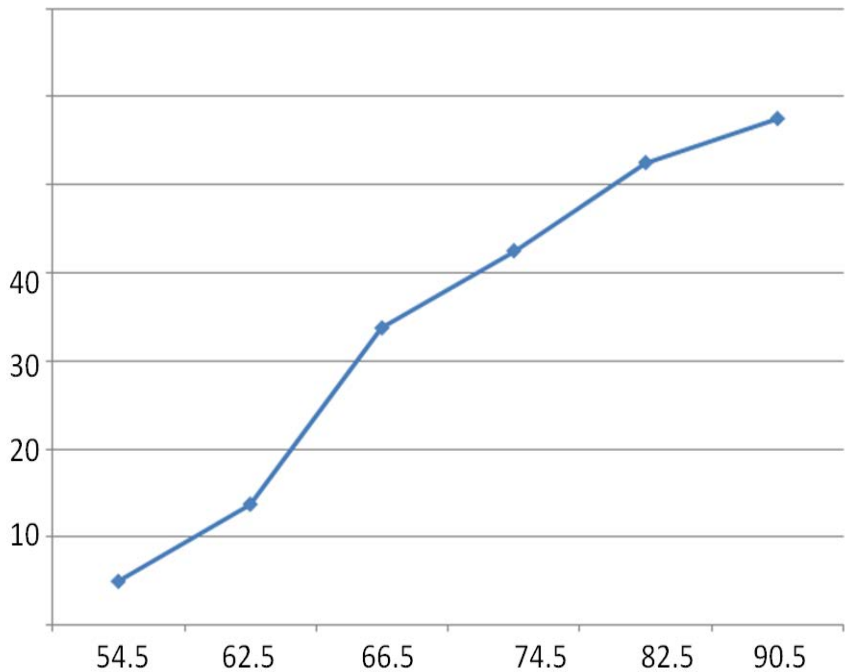


- Cumulative Frequency Graphs: Another way of representing continuous data is to draw a cumulative frequency graph. The cumulative frequencies are plotted against the upper class boundaries of the corresponding class

- Example: The grouped frequency distribution in the following table summarizes the masses in kilograms, measured to the nearest kilogram, of a sample of 38 students. Represent the data using a cumulative frequency graph

Mass (g)	Class Boundaries	Frequency	Cumulative Frequency
47 - 54	46.5 – 54.5	4	4
55 - 62	54.5 – 62.5	7	11
63 - 66	62.5 – 66.5	8	19
67 - 74	66.5 – 74.5	7	26
75 - 82	74.5 – 82.5	8	34
83 - 90	82.5 – 900.5	4	38

•



- We can use the cumulative frequency graph to get other information also. For example, what proportion of the students has mass less than 60 kg?

  From the graph the frequency is about 8.8.The proportion under 60 kg = 8.8/38 = 0.23 or 23%

- Pie Charts: Pie charts are useful for representing percentage allocation data.
  For example, the budget of a household can be represented effectively through a pie chart.
  Pie charts can be thought of as **circle graphs**. To calculate the quantities represented by each slice of the pie chart we need to take the angular fraction of the given total. i.e. If a particular frequency is x, and n is the total number, then the allocation angle for that class will be $(x/n).360$

- Example: In a survey, 90 people were asked to indicate which one of five musical instruments they played. The information is given in the following table.

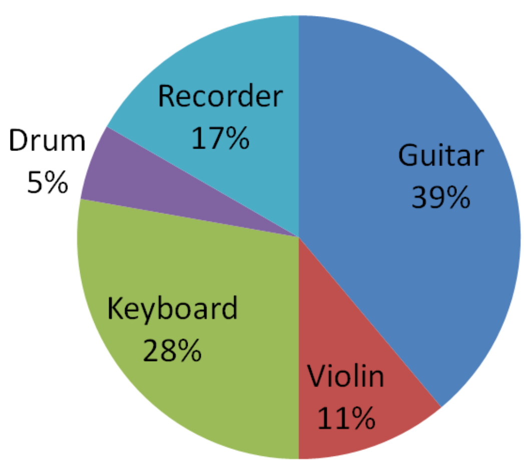
Instrument	Number of People
Guitar	35
Violin	10
Keyboard	25
Drum	5
Recorder	15

Represent the data on a pie chart

Solution: We first calculate the allocation angle for each group:

Instrument	Number of People	Allocation Angle	Percentage
Guitar	35	140^o	39%
Violin	10	40^o	11%
Keyboard	25	100^o	28%
Drum	5	20^o	5%
Recorder	15	60^o	17%
TOTAL	90	360^o	100%

People and Musical Instruments they Play



Lecture 38-39, part of 40

Measures of Central Tendency

- Focus of Statistics:
  We are concerned in statistics with three things:
  - **Measures of Location**: where the data is clustered or centered. Also called measures of central tendency.
  - **Measures of Dispersion**: How the data is spread out from the center.
  - **Measures of Shape**: How the data sways and peaks.

- Measures of Central Tendency: The **central tendency** is measured by averages. These describe the point about which the various observed values cluster. In mathematics, an **average**, or **central tendency** of a data set refers to a measure of the "middle" or "expected" value of the data set.

- We will be studying the following measures of central tendency: Arithmetic Mean, Geometric Mean, Weighted Mean, Harmonic Mean, Median, and Mode.

- Definition: The **Arithmetic Mean** is the sum of the observation values divided by the number of observations. It is the most widely used measure of central tendency, and is the *only measure* where the *sum of the deviations* of each value from the mean is always *zero*. The formula for calculating the arithmetic mean of n values is: $\bar{x} = \dfrac{x_1 + x_2 + x_3 + \ldots\ldots\ldots\ldots + x_n}{n}$ or, $\bar{x} = \dfrac{\sum_{i=1}^{n} x_i}{n}$.

- Arithmetic Mean of Grouped Data: If $z_1, z_2, z_3, \ldots\ldots\ldots\ldots, z_k$ are the mid-values and $f_1, f_2, f_3, \ldots\ldots\ldots, f_k$ are the corresponding frequencies, where the subscript 'k' stands for the number of classes, then the mean is $\bar{z} = \dfrac{\sum f_i z_i}{\sum f_i}$.

- Example: The math exam scores of 9 students are given below: 59, 66, 65, 74, 71, 67, 71, 62, 77.
  The mean score is: $\bar{x} = \dfrac{59+66+65+74+71+67+71+62+77}{9} = \dfrac{612}{9} = 68$

- Definition: The **Geometric Mean** is often used when calculating investment returns over multiple periods, or to find a compound growth rate. It is computed by taking the *nth* root of the product of n values. In general, when we are dealing with numbers that need to be multiplied, the geometric mean gives a more realistic picture than the arithmetic mean. The geometric mean for n values is:
  $GM = \sqrt[n]{a_1 a_2 \ldots a_n} = \left(\prod_{i=1}^{n} a_i\right)^{n}$

- Geometric Mean of Grouped Data: If the "*n*" non-zero and positive variable values $x_1, x_2, ..., x_n$ occur $f_1, f_2, ..., f_n$ times, respectively, then the geometric mean of the set of observations is defined by: $GM = \left[x_1^{f_1}\ x_2^{f_2} \cdots\cdots\cdots x_n^{f_n}\right]^{\frac{1}{N}} = \left[\prod_{i=1}^{n} x_i^{f_i}\right]^{\frac{1}{N}}$ where $N = \sum_{i=1}^{n} f_i$

- Example (a): Company A has grown over the last 3 years by 10 million, 12 million and 14 million dollars. What is the average annual growth amount?

  This is arithmetic mean = (10 + 12 + 14)/3 = 12 million dollars.

- Example (b): The profit of Company B has grown over last three years by 2.5%, 3%, and 3.5%. What is the average growth rate?

  This is geometric mean = $[(1.025)(1.030)(1.035)]^{1/3} = 2.9992\%$

- Definition**:** The **Weighted mean** is a special case of the mean that allows different weights on different observations. Formally, the weighted mean of a non-empty set of data, $x_1, x_2, \ldots, x_n$, with non-negative weights $w_1, w_2, \ldots, w_n$, is the quantity calculated by $\bar{x} = \frac{w_1 x_1 + w_2 x_2 + \ldots w_n x_n}{w_1 + w_2 + \cdots w_n}$

  Note: The arithmetic mean is weighted mean where all the weights equal 1

- Example: Given two school classes, one with 20 students, and one with 30 students, the grades in each class on a test were:

  Morning class = 62, 67, 71, 74, 76, 77, 78, 79, 79, 80, 80, 81, 81, 82, 83, 84, 86, 89, 93, 98

  Afternoon class = 81, 82, 83, 84, 85, 86, 87, 87, 88, 88, 89, 89, 89, 90, 90, 90, 90, 91, 91, 91, 92, 92, 93, 93, 94, 95, 96, 97, 98, 99

  What is the average score of all the students if the average for the morning class is 80 and the average of the afternoon class is 90?

  Solution: The straight average of 80 and 90 is 85, the mean of the two class means. However, this does not account for the difference in number of students in each class, and the value of 85 does not reflect the average student grade (independent of class). The average student grade can be obtained by either averaging all the numbers without regard to classes as $\bar{x} = \frac{4300}{50} = 86$, or weighting the class means by the number of students in each class: $\bar{x} = \frac{20(80) + 30(90)}{20 + 30} = 86$.

- Definition: The **Harmonic mean** is often used by investors to find the average cost of shares purchased over time. In certain situations, especially many situations involving *rates* and *ratios*, the harmonic mean provides the truest average. Formally, for a set of positive data values, $x_1, x_2, \ldots, x_n$, the harmonic mean is the reciprocal of the arithmetic mean of the reciprocals:

$$HM = \frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}} = \frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}}.$$

- Example (c)**:** You go on a 100 km trip. Suppose you travel half the **time** at 40 km/h and half the time at 60 km/h. What is the average speed?

  This is Arithmetic mean = (40 + 60)/2 = 50 km/hr

- Example (d): You go on a 100 km trip. Suppose you travel half the **distance** of your trip at 40 km/h, and the remaining half at 60 km/h**.** What is the average speed?

  This is Harmonic mean = 2/(1/40 + 1/60) = 48 km/hr

- Definition: The **median** is the middle value of the observations such that the number of observations above it is equal to the number of observations below it. If the number of values, *n* is odd, then the median is the middle value, i.e. $M_e = x_{\frac{1}{2}(n+1)}$; if *n* is even, then the median is the average of the middle two values, i.e. $M_e = \frac{1}{2}\left(x_{\frac{n}{2}} + x_{\frac{n}{2}+1}\right)$

- Median of Grouped Data: For grouped data the median is given by: $M_e = L_o + \frac{h}{f_o}\left(\frac{n}{2} - F\right)$, where
  - L_0 = Lower class boundary of the median class
  - h = Width of the median class
  - f_0 = Frequency of the median class
  - F = Cumulative frequency of the pre-median class

- Steps to Find the Median of Grouped Data:
  - Compute the less than type cumulative frequencies.
  - Determine n/2 , one-half of the total number of cases.
  - Locate the median class for which the cumulative frequency is more than N/2 .
  - Determine the lower limit of the median class. This is L_0.
  - Sum the frequencies of all classes prior to the median class. This is *F*.
  - Determine the frequency of the median class. This is f_0.
  - Determine the class width of the median class. This is *h*.
  - Apply the Formula.

- Definition: **Mode** is the value of a distribution for which the frequency is maximum. In other words, mode is the value of a variable, which occurs with the highest frequency.

- Example: The mode of the list (1, 2, 2, 3, 3, 3, 4) is 3. The mode is not necessarily well defined. The list (1, 2, 2, 3, 3, 5) has the two modes 2 and 3.

- Mode of Grouped Data: The formula for finding mode from grouped data is

$$M_0 = L_1 + \frac{\Delta_1}{\Delta_1 + \Delta_2} h, \text{ where}$$

  - L_1 = Lower boundary of modal class
  - Δ_1 = difference of frequency between modal class and class before it
  - Δ_2 = difference of frequency between modal class and class after
  - h = class interval

- Example: A commuter who travels to work by car has a choice of two different routes, V and W. He decides to compare his journey times for each route. So, he records the journey times, in minutes, for 10 consecutive working days, for each route. The results are:

Route V	53	52	48	51	49	47	42	48	57	53
Route W	43	41	39	108	52	42	38	45	39	51

  - Calculate the means and medians for the two routes respectively.
  - Which average do you think is more suitable for comparing the time taken on each route?

- Solution: For Route V: Mean = (53+52+48+51+49+47+42+48+57+53)/10 = 500/10 = 50 Arranging in ascending order, we get: 42, 47, 48, 48, 49, 51, 52, 53, 53, 57. So, the Median is (49 + 51)/2 = 50

  For Route W: Mean = (43+41+39+108+52+42+38+45+39+51)/10 = 498/10 = 49.8. Arranging in ascending order, we get: 38, 39, 39, 41, 42, 43, 45, 51, 52,108. So, the Median is (42 + 43)/2 = 42.5

  Route W is quicker.

- Example: Consider the two sets of data A and B

  A: 48, 52, 60, 60, 60, 68, 72

  B: 0, 10, 60, 60, 60, 110, 120

  For both sets of data the mean = median = mode = 60. Are the two data sets the same?

- If you were given nothing but the measures of location, you might be tempted to think that the two sets of data are similar. But B is much more spread out than A. It is necessary for us to devise some new measures to summarize the **spread of data**.