MTH101 Practice Questions/ Solutions Lecture No. 23 to 45

Lecture No. 23: Maximum and Minimum Values of Functions

Q 1: Find the minimum value of the function $f(x) = x^3 - 27x + 4$ attains in the interval [4,-4].

Solution:

First of all we have to find critical points by putting $f'(x)$=0

$f(x) = x^3 - 27x + 4$

$f'(x) = 3x^2 - 27$

$put\ f'(x) = 0$

$\Rightarrow 3x^2 - 27 = 0$

$\Rightarrow 3(x^2 - 9) = 0$

$\Rightarrow x^2 - 9 = 0 \Rightarrow x = -3, 3$

now we have points 4,-4, -3 and 3 we will check on all these points

$f(4) = (4)^3 - 27(4) + 4 = -40$

$f(-2) = (-4)^3 - 27(-4) + 4 = 48$

$f(3) = (3)^3 - 27(3) + 4 = -50$

$f(-3) = (-3)^3 - 27(-3) + 4 = 58$

So minimum value $= -50$

Q 2: Find the maximum value of the function $f(x) = x^3 + 3x^2 - 9x$ attains in the interval [-4, 3].

Solution:

First of all we have to find critical points by putting $f'(x) = 0$

$f(x) = x^3 + 3x^2 - 9x$

$f'(x) = 3x^2 + 6x - 9$

$put\ f'(x) = 0$

$\Rightarrow 3x^2 + 6x - 9 = 0$

$\Rightarrow x^2 + 2x - 3 = 0$

$\Rightarrow x^2 + 3x - x - 3 = 0$

$\Rightarrow x(x+3) - (x+3) = 0$

$\Rightarrow (x-1)(x+3) = 0 \ \Rightarrow x = 1, x = -3$

now we have points 1,-3, 3 and -4 we will check on all these points

$f(1) = (1)^3 + 3(1)^2 - 9(1) = -5$

$f(-3) = (-3)^3 + 3(-3)^2 - 9(-3) = 27$

$f(3) = (3)^3 + 3(3)^2 - 9(3) = 27$

$f(-4) = (-4)^3 + 3(-4)^2 - 9(-4) = 20$

Hence maximum value = $f(-3) = f(3) = 27$

Q 3: Find the absolute maximum and absolute minimum values of the function $f(x) = 4 - x^2$ on interval $-3 \le x \le 1$.

Solution:

First we will find critical points:

put $f^{/}(x) = 0$

$\Rightarrow -2x = 0$

$\Rightarrow x = 0$

Now we find value of $f(x)$ at critical point and at the end points of interval :

$f(0) = 4 - (0)^2 = 4$

$f(1) = 4 - (1)^2 = 3$

$f(-3) = 4 - (-3)^2 = -5$

Hence absolute maximum = 4 and absolute minimum=-5

Q 4: Find the absolute maximum and absolute minimum values of the function $f(x) = 2 + x$ on interval $-2 \le x \le 2$.

Solution:

First we will find critical points:

Since $f^{/}(\mathrm{x}) = 1$, so there is no critical points.

Now we find value of $f(x)$ at the end points of interval :

$f(-2) = 2 - 2 = 0$

$f(2) = 2 + 2 = 4$

Hence absolute maximum = 4 and absolute minimum=0

Q 5: Find the maximum and minimum value of the function $f(x) = 3x^4 - 24x^2 + 1$ on the interval $(-\infty, +\infty)$.

Solution:

This is a continuous function on the given interval and

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} (3x^4 - 24x^2 + 1) = +\infty$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} (3x^4 - 24x^2 + 1) = +\infty$$

So f has a minimum but no maximum value in the interval $(-\infty, +\infty)$. To find the minimum value put $f^{/}(x) = 0$ i.e

$f'(x) = 12x^3 - 48x = 0 \Rightarrow 12x(x^2 - 4) = 0 \Rightarrow x = 0 \; and \; x = \pm 2$ are the critical points.

At $x = 0$, $f(0) = 1$,

at $x = 2$, $f(2) = 3(2)^4 - 24(2)^2 + 1 = -47$ and

at $x = -2$, $f(-2) = 3(-2)^4 - 24(-2)^2 + 1 = -47$

so minimum value occurs at $x = \pm 2$ and it is equal to '$f(x) = -47$'

Lecture No. 24: Newton's Method, Rolle's Theorem and Mean Value Theorem
Lecture No. 25: Integrations

Q 1: Check the validity of Mean Value Theorem for $f(x)=3x-x^3$ on the interval$[0,2]$. Also find 'c' if possible

Solution: Because f is a polynomial so continuous and differentiable everywhere hence on$[0,2]$. As hypothesis of the mean value theorem is satisfied so we can

Find a 'c' such that

$$\frac{f(b)-f(a)}{b-a}=f'(c) \qquad where \quad f(a)=f(0)=0 \quad \text{and} \quad f(b)=f(2)=-2$$

$$f'(x)=3-3x^2$$

$$f'(c)=3-3c^2$$

$$\frac{-2-0}{2-0}=3-3c^2$$

$$3-3c^2=-1$$

$$\Rightarrow 3(1-c^2)=-1$$

$$\Rightarrow 1-c^2=-\frac{1}{3}$$

$$\Rightarrow c^2=\frac{4}{3}$$

$$\Rightarrow c=\pm\frac{2}{\sqrt{3}}$$

As $c=\frac{2}{\sqrt{3}}\in(0,2)$ and $c=-\frac{2}{\sqrt{3}}\notin(0,2)$ so value of $c=\frac{2}{\sqrt{3}}$.

Q 2: To estimate the solution of the equation $x^3-7x-1=0$ by using the newton's method if we start with $x_1=2$ then findx_2.

Solution:

$$f(x)=x^3-7x-1,$$

$$f'(x)=3x^2-7,$$

$$x_1=2\Rightarrow f(2)=(2)^3-7(2)-1=-7,$$

$$f'(2)=3(2)^2-7=5.$$

By Newton's Method $\quad x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)} \quad$ where $n=1,\ 2,\ 3,\ .....,$

So, for $n=1$, $\quad x_2=x_1-\frac{f(x_1)}{f'(x_1)}=2-\frac{(-7)}{5}=\frac{17}{5}.$

Q 3: Evaluate the following integrals.

I. $\int \sec x(\sec x + \cos x)dx$

II. $\int \frac{5\sin x - 3\cos^2 x}{\cos^2 x}dx$

Solution:

(I) $\int \sec x(\sec x + \cos x)dx$

$$= \int (\sec^2 x + \sec x \cos x)dx$$

$$= \int \sec^2 x dx + \int \sec x \cos x dx$$

$$= \tan x + \int \frac{\cos x}{\cos x} dx$$

$$= \tan x + \int dx$$

$$= \tan x + x + c$$

(II) $\int \frac{5\sin x - 3\cos^2 x}{\cos^2 x}dx$

$$= \int \left[5\frac{\sin x}{\text{cosxcosx}} - 3 \right] dx$$

$$= \int 5\frac{\sin x}{\cos x}\frac{1}{\cos x} dx - 3\int dx$$

$$= \int 5\tan x \sec x dx - 3\int dx$$

$$= 5\sec x - 3x + c$$

Q 4: Given that $f'(x) = 16x^3 - 4x$, find $f(3)$.

Solution:

First of all, we need to find the function f(x) for which the derivative is given. For that we will integral the derivative function which gives

$$f(x) = \int f'(x)dx$$

$$= \int (16x^3 - 4x)dx$$

$$= \frac{16x^4}{4} - \frac{4x^2}{2} + c$$

$$f(x) = 4x^4 - 2x^2 + c$$

$$f(3) = 4(3)^4 - 2(3)^2 + c$$

$$f(3) = 306 + c$$

Lecture No. 26: Integration by Substitution
Lecture No. 27: Sigma Notation
Lecture No. 28: Area as Limit

Q 1: Evaluate the integral by using substitution method: $\int \frac{1}{t^2}\cos\left(\frac{1}{t}-1\right)dt$.

Solution:

Let $u = \frac{1}{t}-1,$

$\Rightarrow \quad du = -\frac{1}{t^2}dt \quad \Rightarrow \frac{1}{t^2}dt = -du,$

$$\therefore \int \frac{1}{t^2}\cos\left(\frac{1}{t}-1\right)dt = \int \cos u\,(-du),$$
$$= -\sin u + C,$$
$$= -\sin\left(\frac{1}{t}-1\right)+C. \qquad (\because \text{by replacing with the original value})$$

Q 2: Evaluate the indefinite integral by substitution method: $\int \frac{(1+\ln x)^3}{x}dx$.

Solution:

Let $t = 1+\ln x,$

$\Rightarrow \quad dt = \frac{1}{x}dx,$

Hence, $\int \frac{(1+\ln x)^3}{x}dx = \int t^3 dt = \frac{t^4}{4}+C = \frac{(1+\ln x)^4}{4}+C.$

Q 3: Evaluate the sum: $\sum_{k=1}^{7}(k^2-6).$

Solution:

$$\sum_{k=1}^{7}(k^2-6) = \sum_{k=1}^{7}k^2 - \sum_{k=1}^{7}6\,,$$
$$= \frac{7(7+1)(2(7)+1)}{6} - 6(7),$$
$$= \frac{840}{6} - 42,$$
$$= 98.$$

Q 4: Express $\sum_{k=2}^{5} 3^{k-2}$ in sigma notation so that the lower limit is '0' rather than '2'.

Solution:

We will define a new summation index 'j' by the relation

$j = k - 2 \text{ or } j + 2 = k,$

Now when $k = 2, j = 2 - 2 = 0,$

When $k = 5, j = 5 - 2 = 3,$

So the new summation will become $\sum_{j=0}^{3} 3^{j+2-2} = \sum_{j=0}^{3} 3^{j}$.

Q 5: Find the area of the k$^{\text{th}}$ rectangle below the curve $y = x^2$ on the interval $[0, 2]$ by taking x_k^* as right end point and left end point.

Solution:

In order to find the area of k$^{\text{th}}$ rectangle, first of all we will find the width or base of the rectangle that is Δx.

$$\Delta x = \frac{b-a}{n} = \frac{2}{n}.$$

Right end point

$x_k^* = a + k\Delta x,$

$= 0 + k.\frac{2}{n} = \frac{2k}{n}.$

The height of the rectangle is

$f(x_k^*) = \left(\frac{2k}{n}\right)^2 = \frac{4k^2}{n^2}.$

Thus, the area of the k$^{\text{th}}$ rectangle will be

Area=height×base

$= f(x_k^*).\Delta x,$

$= \frac{4k^2}{n^2}.\frac{2}{n} = \frac{8k^2}{n^3}.$

Left end point

$x_k^* = a + (k-1)\Delta x,$

$= 0 + (k-1).\frac{2}{n} = \frac{2(k-1)}{n}.$

The height of the rectangle is

$$f(x_k^*) = \left(\frac{2(k-1)}{n}\right)^2 = \frac{4(k-1)^2}{n^2}.$$

Thus, the area of the k$^{\text{th}}$ rectangle will be

Area=height×base

$$= f(x_k^*).\Delta x,$$

$$= \frac{4(k-1)^2}{n^2}.\frac{2}{n} = \frac{8(k-1)^2}{n^3}.$$

Q 6: Find the approximate area under the graph of function $y = x$ over the interval $[0, 2]$ by taking $\Delta x = \frac{2}{n}$ and $x_k^* = \frac{2k}{n}$.

Solution:

Given that $\Delta x = \frac{2}{n}$ and $x_k^* = \frac{2k}{n}$,

$$f(x_k^*) = \frac{2k}{n},$$

$$f(x_k^*).\Delta x = \frac{2k}{n}.\frac{2}{n} = \frac{4k}{n^2},$$

$$\sum_{k=1}^{n} f(x_k^*).\Delta x = \sum_{k=1}^{n} \frac{4k}{n^2},$$

$$= \frac{4}{n^2}\sum_{k=1}^{n} k,$$

$$= \frac{4}{n^2}\frac{n(n+1)}{2} = \frac{2(n+1)}{n} = 2\left(1+\frac{1}{n}\right).$$

$$\text{Area} = \lim_{n\to\infty}\left(\sum_{k=1}^{n} f(x_k^*).\Delta x\right) = \lim_{n\to\infty} 2\left(1+\frac{1}{n}\right) = 2.$$

Lecture No. 29: Definite Integral
Lecture No. 30: First Fundamental Theorem of Calculus
Lecture No. 31: Evaluating Definite Integral by Substitution

Q 1: Find $\int_2^5 f(x)dx$ if $\int_3^2 f(x)dx = -5$, $\int_3^4 f(x)dx = 3$, $\int_5^4 f(x)dx = 4$

Solution:

$$\int_5^4 f(x)dx = 4 \Rightarrow \int_4^5 f(x)dx = -4 \text{ and } \int_3^2 f(x)dx = -5 \Rightarrow \int_2^3 f(x)dx = 5$$

$$\therefore \int_2^5 f(x)dx = \int_2^3 f(x)dx + \int_3^4 f(x)dx + \int_4^5 f(x)dx$$

$$= 5 + 3 - 4 = 4$$

Q 2: Express the area of the region below the line $7x + 2y = 25$, above x-axis and between the lines $x = 0$, $x = 4$ as a definite integral. Also express this integral as a limit of the Riemann Sum.

Solution:

$$7x + 2y = 25$$

$$2y = 25 - 7x \Rightarrow y = \frac{25 - 7x}{2}$$

$$\int_0^4 \frac{25 - 7x}{2} dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \left(\frac{25 - 7x_k^*}{2} \right) \Delta x_k$$

Q 3: Evaluate the integral $\int_1^3 \frac{x^3 - 1}{x - 1} dx$.

Solution:

$$\int_1^3 \frac{x^3 - 1}{x - 1} dx = \int_1^3 \frac{(x-1)(x^2 + x + 1)}{x - 1} dx$$

$$= \int_1^3 (x^2 + x + 1) dx$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + x \Big|_1^3$$

$$= \left(\frac{27}{3} + \frac{9}{2} + 3 \right) - \left(\frac{1}{3} + \frac{1}{2} + 1 \right) = \left(12 + \frac{9}{2} \right) - \left(\frac{2 + 3 + 6}{6} \right)$$

$$= \left(\frac{24 + 9}{2} \right) - \frac{11}{6} = \frac{33}{2} - \frac{11}{6} = \frac{99 - 11}{6} = \frac{88}{6} = \frac{44}{3}$$

Q 4: Evaluate the definite integral $\int_1^4 \left(3+3\sqrt{x}\right)dx$.

Solution:

$$\int_1^4 \left(3+3\sqrt{x}\right)dx = 3x + 3\frac{x^{3/2}}{3/2}\Bigg|_1^4$$

$$= 3x + 2x^{3/2}\Big|_1^4$$

$$= \left(3(4)+2(4)^{3/2}\right) - \left(3(1)+2(1)^{3/2}\right) = \left(12+2(8)\right) - \left(3+2\right) = 28-5 = 23$$

Q 5: Evaluate the integral $\int \frac{1}{\sqrt{x}.(2+\sqrt{x})^3}\,dx$ by using proper substitution.

Solution:

Let $u = 2+\sqrt{x}$

$$\Rightarrow du = \frac{1}{2\sqrt{x}}\,dx$$

$$\Rightarrow 2du = \frac{1}{\sqrt{x}}\,dx$$

Putting these values in the given integral, it becomes

$$\int \frac{1}{\sqrt{x}.(2+\sqrt{x})^3}\,dx = 2\int \frac{1}{u^3}\,du =$$

$$= 2\int u^{-3}\,du$$

$$= 2\left|\frac{u^{-3+1}}{-3+1}\right| = \frac{2}{-2u^2}$$

$$= \frac{-1}{\left(2+\sqrt{x}\right)^2} + c$$

Q 6: Use the Substitution method to express the following definite integrals in terms of the variable 'u' but do not evaluate the integrals.

i. $\int_0^{\frac{\pi}{2}} e^{\sin t} \cos t \; dt$

ii. $\int_0^1 4t\sqrt{1-t^2} \; dt$

Solution:

i.

Let $u = \sin t$

$\Rightarrow du = \cos t \, dt$

$t = 0 \Rightarrow u = \sin 0 = 0$

and $t = \frac{\pi}{2} \Rightarrow u = \sin\frac{\pi}{2} = 1$

as t goes from 0 to $\frac{\pi}{2}$

so u goes from 0 to 1

$$\int_0^{\frac{\pi}{2}} e^{\sin t} \cos t \; dt = \int_0^1 e^u \; (du) = \int_0^1 e^u du$$

ii.

Let

$u = 1 - t^2$

$\Rightarrow du = -2t dt$

$-2du = 4tdt$

$t = 1 \Rightarrow u = 1-(1)^2 = 0$

and $t = 0 \Rightarrow u = 1-(0)^2 = 1$

as t goes from 0 to 1

so u goes from 1 to 0

$$\int_0^1 4t\sqrt{1-t^2} \; dt = -2\int_1^0 \sqrt{u} \; du = 2\int_0^1 \sqrt{u} \; du \quad \begin{pmatrix} \because -2tdt = du \\ 4tdt = -2du \end{pmatrix}$$

Lecture No. 32: Second Fundamental Theorem of Calculus
Lecture No. 33: Application of Definite Integral
Lecture No. 34: Volume by slicing; Disks and Washers

Q 1: Find a definite integral indicating the area enclosed by the curves $y = x^2 - 6x - 7$ and $y = x + 1$. But do not evaluate it further.

Solution:

Here $y = x^2 - 6x - 7$ ______ (1),

and $y = x + 1$ ______ (2),

equating (1) and (2)

$$\Rightarrow \quad x^2 - 6x - 7 = x + 1,$$
$$x^2 - 6x - 7 - x - 1 = 0,$$
$$x^2 - 7x - 8 = 0,$$
$$x^2 - 8x + x - 8 = 0,$$
$$x(x-8) + 1(x-8) = 0,$$
$$\therefore \quad (x-8)(x+1) = 0,$$
$$\Rightarrow \quad x = -1 \text{ and } x = 8,$$

as $x + 1 \geq x^2 - 6x - 7$ for $-1 \leq x \leq 8$,

$$\therefore \quad \int_{-1}^{8} \left((x+1) - (x^2 - 6x - 7)\right) dx.$$

Q 2: Find the area of the region enclosed by the curves $y = x^3$ and $y = x$ between $x = 0$ and $x = \frac{1}{3}$.

Solution:

Since we have $y = x^3$ and $y = x$

$$\Rightarrow \quad x \geq x^3 \text{ in the interval } \left[0, \frac{1}{3}\right].$$

Thus required area is $A = \int_0^{\frac{1}{3}} \left(x - x^3\right) dx = \left.\frac{x^2}{2} - \frac{x^4}{4}\right|_0^{\frac{1}{3}}$

$$= \frac{1}{2}\left(\frac{1}{3}\right)^2 - \frac{1}{4}\left(\frac{1}{3}\right)^4 = \frac{1}{18} - \frac{1}{324}$$
$$= \frac{17}{324}$$

Q 3: Evaluate $\int_{\pi}^{0} \frac{1+\sin 2t}{2} dt$.

Solution:

$$\int_{\pi}^{0} \frac{1+\sin 2t}{2} dt = \int_{\pi}^{0} \left(\frac{1}{2} + \frac{\sin 2t}{2} \right) dt$$

$$= \int_{\pi}^{0} \frac{1}{2} dt + \int_{\pi}^{0} \frac{\sin 2t}{2} dt$$

$$= \frac{1}{2} \left| t \right|_{\pi}^{0} - \frac{1}{4} \left| \cos 2t \right|_{\pi}^{0}$$

$$= \frac{1}{2}(0-\pi) - \frac{1}{4}(\cos 0 - \cos 2\pi)$$

$$= -\frac{\pi}{2} - \frac{1}{4}(1-1) = -\frac{\pi}{2}$$

Q 4: Evaluate $\frac{d}{dx}\left[\int_{1}^{x} \cos t dt \right]$ by using second fundamental theorem of calculus. Also Check the validity of this theorem.

Solution:

Since $f(x) = \cos x$ is continuous on $(1, x)$, so by second fundamental theorem of calculus

$$\frac{d}{dx}\left[\int_{1}^{x} \cos t dt \right] = \cos x$$

Now we will verify this result by evaluating this integral.

$$\int_{1}^{x} \cos t dt = \left| \sin t \right|_{1}^{x} = \sin x - \sin 1$$

$$\Rightarrow \frac{d}{dx} \int_{1}^{x} \cos t dt = \frac{d}{dx}(\sin x - \sin 1) = \cos x = f(x)$$

Q 5: Find the volume of a solid bounded by two parallel planes perpendicular to the x-axis at $x = 0$ and $x = 3$, where the cross section perpendicular to the x- axis is a rectangle having dimensions 'a' and 'b'

Solution:

Here dimensions are $a \,\&\, b \quad \Rightarrow A = ab$

$$V = \int_0^3 ab\,dx \qquad \left(\because V = \int_c^d A(x)\ dx\right)$$

$$= ab\int_0^3 dx \Rightarrow ab.x\Big|_0^3 \Rightarrow 3ab$$

Q 6: Find the volume of the solid generated when the region between the graphs of $f(x) = \frac{1}{2} + x$ and $g(x) = x$ over the interval [0,2] is revolved about the x-axis.

Solution:

Since the volume enclosed between two curves $y = f\left(x\right)$ *and* $y = g\left(x\right)$ revolve about the x-axis so

$$V = \int_a^b \pi\left(g^2\left(x\right) - f^2\left(x\right)\right)dx$$

Here $f\left(x\right) = \frac{1}{2} + x$ *and* $g\left(x\right) = x$ *as* $f(x) \geq g(x)$ *for* $0 \leq x \leq 2$

$a = 0$, $b = 2$, so

$$V = \int_0^2 \pi\left((\frac{1}{2} + x)^2 - x^2\right)dx$$

$$V = \int_0^2 \pi(\frac{1}{4} + x^2 + x - x^2)dx$$

$$= \int_0^2 \pi(\frac{1}{4} + x)dx \Rightarrow \pi\left[\frac{x}{4} + \frac{x^2}{2}\right]_0^2 \Rightarrow \frac{5\pi}{2}$$

Lecture No. 35: Volume by Cylindrical Shells
Lecture No. 36: Length of Plane Curves
Lecture No. 37: Area of Surface of Revolution

Q 1: Use cylindrical shells to find the volume of the solid generated when the region 'R' in the first quadrant enclosed between $y = 4x$ and $y = x^3$ is revolved about the y-axis.

Solution:

To find the limit of integration, we will find the point of intersection between two curves.

Equating $y = 4x$ and $y = x^3$, we get

$$x^3 = 4x,$$
$$\Rightarrow x^3 - 4x = 0,$$
$$\Rightarrow x(x^2 - 4) = 0,$$
$$\Rightarrow x = 0, x = \pm 2.$$

Since our region R is in first quadrant so we ignore $x = -2$. Hence limit of integration is $x = 0$ to $x = 2$.

So $V = \int_0^2 2\pi x(4x - x^3)dx,$

$$= 2\pi \int_0^2 (4x^2 - x^4)dx,$$

$$= 2\pi \left| 4\frac{x^3}{3} - \frac{x^5}{5} \right|_0^2.$$

After simplification, we get

$$V = \frac{128}{15}\pi.$$

Q 2: Use cylindrical shell method to find the volume of the solid generated when the region enclosed between $y = x^3$ and the x-axis in the interval $[0,3]$ is revolved about the y-axis.

Solution:

$$V = \int_0^3 2\pi\, x(x^3)\, dx = 2\pi \int_0^3 x^4\, dx,$$

$\left(\because \text{Here } f(x) = x^3, \quad a = 0;\ b = 3. \quad \text{Cylindrical shells revolved about the } y\text{-axis}: V = \int_a^b 2\pi\, x\, f(x)\, dx \right)$

$$= 2\pi . \frac{1}{5} \left| x^5 \right|_0^3.$$

After simplification, we get $V = \dfrac{486}{5}\pi.$

Q 3: Find the arc length of the curve $y=\frac{2}{3}(x-1)^{\frac{3}{2}}$ from $x=0$ to $x=\frac{1}{2}$.

Solution:

$$\because \quad y=\frac{2}{3}(x-1)^{\frac{3}{2}},$$

$$\Rightarrow \frac{dy}{dx}=\frac{2}{3}\left(\frac{3}{2}\right)(x-1)^{\frac{1}{2}}=(x-1)^{\frac{1}{2}}.$$

$$\because \quad L=\int_a^b \sqrt{1+\left[f'(x)\right]^2}\,dx,$$

$$\Rightarrow L=\int_0^{\frac{1}{2}} \sqrt{1+\left[(x-1)^{\frac{1}{2}}\right]^2}\,dx=\int_0^{\frac{1}{2}} \sqrt{(1+x-1)}\,dx=\int_0^{\frac{1}{2}} \sqrt{x}\,dx,$$

$$=\frac{2}{3}x^{\frac{3}{2}}\Big|_0^{\frac{1}{2}},$$

$$=\frac{2}{3}\left(\frac{1}{2}\right)^{\frac{3}{2}},$$

$$=\frac{1}{3\sqrt{2}}.$$

Q 4: If f is a smooth function on $[1,4]$, then find a definite integral indicating the arc length of the curve $x=y^{\frac{2}{3}}$ from $y=1$ to $y=4$. **Note**: Do not evaluate it further.

Solution:

Here $\quad x=g(y)=y^{\frac{2}{3}},$

$$\Rightarrow \qquad g'(y)=\frac{2}{3}y^{-\frac{1}{3}}.$$

$\because$ Arc length $\quad L=\int_a^b \sqrt{1+\left[g'(y)\right]^2}\;dy,$

$$\therefore \qquad L=\int_1^4 \sqrt{1+\frac{4}{9}y^{\frac{-2}{3}}}\;dy.$$

Q 5: Find the area of the surface generated by revolving the curve $y=\sqrt{1-x^2}$; $0\le x\le 1$ about the $x-$axis.

Solution:

$\because\ f(x)=y=\sqrt{1-x^2},$

$\Rightarrow\ f'(x)=-\dfrac{x}{\sqrt{1-x^2}}.$

$\because\ \ S=\int_a^b 2\pi f(x)\sqrt{1+\left[f'(x)\right]^2}\,dx,$

$$\Rightarrow S=\int_0^1 2\pi\sqrt{1-x^2}\sqrt{1+\left[-\frac{x}{\sqrt{1-x^2}}\right]^2}\,dx,$$

$$=\int_0^1 2\pi\sqrt{1-x^2}\sqrt{1+\frac{x^2}{1-x^2}}\,dx,$$

$$=\int_0^1 2\pi\sqrt{1-x^2}\sqrt{\frac{1-x^2+x^2}{1-x^2}}\,dx,$$

$$=\int_0^1 2\pi\,dx,$$

$$=2\pi x\Big|_0^1,$$

$$=2\pi.$$

Q 6: Write a definite integral indicating the area of the surface generated by revolving the curve $y=x^2$; $0\le x\le 2$ about the $x-$axis. **Note**: Do not evaluate it further.

Solution:

$\because\ \ y=f(x)=x^2\ ;\ \ 0\le x\le 2,$

$\Rightarrow \dfrac{dy}{dx}=f'(x)=2x.$

$$\therefore\ \ S=\int_0^2 2\pi\left(x^2\right)\sqrt{1+\left(2x\right)^2}\,dx,\qquad \left(\because S=\int_c^d 2\pi\ f(x)\sqrt{1+\left[f'(x)\right]^2}\ dx\right)$$

$$=2\pi\int_0^2 x^2\sqrt{1+4x^2}\,dx.$$

Lecture No. 38: Work and Definite Integral
Lecture No. 39: Improper Integral
Lecture No. 40: L'Hopital's Rule

Q 1: Find the work done by the force 500*x* if an object moves in the positive direction over the interval [0.16, 0.19]?

Solution:
Here
F(x) =500x; [0.16, 0.19]

$W = \int_{0.16}^{0.19} 500x\, dx$ since $W = \int_a^b F(x)\, dx$

$= \left|\frac{500\, x^2}{2}\right|_{0.16}^{0.19}$

$= \frac{500}{2} \left|x^2\right|_{0.16}^{0.19}$

$=250[(0.19)^2-(0.16)^2]$

=2.625 J

Q 2: Find the spring constant if a force took 1800 *J* of work to stretch a spring from its natural length of 5*m* to a length of 8*m*?
Solution:
The work done by F is $W=\int_0^3 F(x)dx$

$1800 = \int_0^3 Kxdx$

$= k\int_0^3 xdx$

$= k\left|\frac{x^2}{2}\right|_0^3$

$= \frac{9k}{2}$

3600=9k

or k=400

Q 3: Evaluate the improper integral:

$$\int_{-\infty}^{0} \frac{1}{(2x+1)^3}\,dx$$

Solution:

$$\int_{-\infty}^{0} \frac{1}{(2x+1)^3}\,dx$$

$$= \lim_{t\to\infty} \int_{t}^{0} \frac{1}{(2x+1)^3}\,dx$$

$$= \lim_{t\to\infty} \frac{1}{2}\int_{t}^{0} \frac{1}{(2x+1)^3}\,(2dx)$$

$$= \lim_{t\to\infty} \frac{1}{2}\int_{t}^{0} (2x+1)^{-3}(2dx)$$

$$= \lim_{t\to\infty} \frac{1}{2}\left|\frac{(2x+1)^{-2}}{-2}\right|_{t}^{0}$$

$$= \lim_{t\to\infty} \frac{-1}{4}\ \left|(2x+1)^{-2}\right|_{t}^{0}$$

$$= \frac{-1}{4}\lim_{t\to\infty}\ [1-(2t+1)^{-2}]$$

$$= \frac{-1}{4}\lim_{t\to\infty}\ [1-\frac{1}{(2t+1)^2}]$$

$$= \frac{-1}{4}[1-0]$$

$$= \frac{-1}{4}$$

Q 4: Solve the improper integral: $\int_1^5 \frac{1}{(x-2)^{\frac{2}{3}}} dx$.

Solution:

$$\int_1^5 \frac{1}{(x-2)^{\frac{2}{3}}} dx$$

$$= \int_1^2 \frac{dx}{(x-2)^{\frac{2}{3}}} + \int_2^5 \frac{dx}{(x-2)^{\frac{2}{3}}}$$

$$= \int_1^2 (x-2)^{-\frac{2}{3}} dx + \int_2^5 (x-2)^{-\frac{2}{3}} dx$$

$$= 3 \left|(x-2)^{\frac{1}{3}}\right|_1^2 \;\; +3 \left|(x-2)^{\frac{1}{3}}\right|_2^5$$

$=3[0- (-1)^{1/3}]+3[(3)^{1/3}-0]$

$=3[(3)^{1/3}-(-1)^{1/3}]$

$=3[\ \sqrt[1/3]{3}\ \ -\ \ \sqrt[1/3]{-1}\ ]$

Q 5: Use L'Hopital's rule to evaluate the limit:

$$\lim_{x\to\frac{\pi}{2}} \frac{\sin x - 1}{Cos2x + 1}$$

Solution:

$$\lim_{x\to\frac{\pi}{2}} \frac{\sin x - 1}{Cos2x + 1} \qquad \frac{0}{0} form$$

By L'Hopital's rule

$$= \lim_{x\to\frac{\pi}{2}} \frac{\cos x}{-2\sin 2x} \qquad \frac{0}{0}\ form$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\cos x}{-2\sin 2x}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\cos x}{-4\sin x \cos x} \qquad since\ \ \sin 2x = 2\sin x \cos x$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{1}{-4\sin x}$$

$$= \frac{-1}{4}$$

Q 6: Evaluate the limit: $\lim_{x\to\infty} \sqrt{x^2 - 5x} - x$.

Solution:

$$\lim_{x\to\infty} \sqrt{x^2 - 5x} - x$$

$$= \lim_{x\to\infty} \frac{\sqrt{x^2 - 5x} - x}{\sqrt{x^2 - 5x} + x} \times \sqrt{x^2 - 5x} + x$$

$$= \lim_{x\to\infty} \frac{(\sqrt{x^2 - 5x})^2 - x^2}{\sqrt{x^2 - 5x} + x}$$

$$= \lim_{x\to\infty} \frac{x^2 - 5x - x^2}{\sqrt{x^2 - 5x} + x}$$

$$= \lim_{x\to\infty} \frac{-5x}{\sqrt{x^2 - 5x} + x}$$

$$= \lim_{x\to\infty} \frac{-5x}{\sqrt{x^2(1 - \frac{5}{x})} + x}$$

$$= \lim_{x\to\infty} \frac{-5x}{x\sqrt{(1 - \frac{5}{x})} + x}$$

$$= \lim_{x\to\infty} \frac{-5x}{x(\sqrt{\left(1 - \frac{5}{x}\right)} + 1)}$$

$$= \lim_{x\to\infty} \frac{-5}{\sqrt{\left(1 - \frac{5}{x}\right)} + 1}$$

$$= \frac{-5}{2}$$

Lecture No. 41: Sequence
Lecture No. 42: Infinite Series
Lecture No. 43: Additional Convergence tests

Q 1: Determine whether the sequence $\{a_n\}$ converges or diverges? Where $a_n = \frac{n+1}{n^2+3}$.

Solution:

Here $a_n = \frac{n+1}{n^2+3}$

We will calculate the limit $\lim_{n\to\infty} \frac{n+1}{n^2+3}$.

Dividing by 'n^2' in denominator and numerator

$$= \lim_{n\to\infty} \frac{\frac{n}{n^2} + \frac{1}{n^2}}{\frac{n^2}{n^2} + \frac{3}{n^2}}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n} + \frac{1}{n^2}}{1 + \frac{3}{n^2}} = \frac{\frac{1}{\infty} + \frac{1}{\infty}}{1 + \frac{3}{\infty}} = \frac{0}{1} = 0$$

The sequence is convergent and its limit is '0'.

Q 2: Determine whether the following sequence is strictly monotone or not. Justify your answer.

$$a_n = \left\{ \frac{2}{n^2} \right\}$$

Solution:

Here

$$a_n = \frac{2}{n^2} \qquad and\ a_{n+1} = \frac{2}{(n+1)^2}$$

$$\frac{a_{n+1}}{a_n} = \frac{2}{(n+1)^2} \times \frac{n^2}{2} = \frac{n^2}{(n+1)^2} < 1$$

$$\frac{a_{n+1}}{a_n} < 1 \Rightarrow a_{n+1} < a_n$$

So this is a decreasing sequence and hence strictly monotonic.

Q 3: Determine whether the sequence $\{a_n\}$ converges or diverges; if it converges then find its limit; where

$$a_n = [\ln 2n - \ln(n+2)].$$

Solution:

$$\lim_{n\to\infty} [\ \ln 2n\ -\ \ln(n+2) = \lim_{n\to\infty} \left[\ln \frac{2n}{n+2}\right]$$

$$= \ln \lim_{n\to\infty} [\ \frac{2n}{n+2}\ ]$$

divide numerator and denominator by n,

$$= \ln \lim_{n\to\infty} [\ \frac{2}{1+\frac{2}{n}}\ ]$$

$$= \ln(2) \qquad \left(\therefore \lim_{n\to\infty} [\ \frac{2}{1+\frac{2}{n}}\ ] = 2\right)$$

The sequence is convergent and its limit is $\ln 2$.

Q 4: Determine whether the following series converges; if so then find the sum:

$$5+\frac{5}{2}+\frac{5}{2^2}+\ldots+\frac{5}{2^{k-1}}+\ldots$$

Solution:

Since $5+\frac{5}{2}+\frac{5}{2^2}+\ldots+\frac{5}{2^{k-1}}+\ldots$ is a geometric series with $a=5$ and $r=\frac{1}{2}$.

And $|r| = \frac{1}{2} < 1$, therefore the series converges and the sum is

$$\frac{a}{1-r} = \frac{5}{1-\frac{1}{2}} \qquad \left(\because \text{ if } |r|<1, \text{the series converges then the sum is } \frac{a}{1-r} = a+ar+ar^2+\ldots ar^{k-1}+\ldots\right)$$

$$= 5\left(\frac{2}{1}\right)$$

$$= 10$$

Q 5: Use the divergence test to show that the given series diverges.

$$\sum_{k=1}^{\infty} \frac{k^2+k+2}{3k^2+1}$$

Solution:

To show that the given series diverges by using divergence test we have to show that

$$\lim_{k\to\infty} \frac{k^2+k+2}{3k^2+1} \neq 0$$

Dividing in numerator and denominator by k^2

$$\lim_{k\to\infty} \frac{(k^2+k+3)/k^2}{(2k^2+1)/k^2} = \lim_{k\to\infty} \frac{1+\frac{1}{k}+\frac{3}{k^2}}{2+\frac{1}{k^2}}$$

$$= \frac{1}{3} \neq 0$$

So according to divergence test the series diverges.

$$\sum_{k=1}^{\infty} \frac{k^2+k+2}{3k^2+1}$$

Q 6: Check the convergence or divergence of the series $\sum_{k=1}^{\infty} \frac{1}{(k+1)!}$ by using the Ratio Test.

Solution:

$$\because \textit{The Ratio Test}: \quad \rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k} \quad \textit{for the series} \sum u_k$$

$$\because \sum_{k=2}^{\infty} \frac{1}{(k+1)!} \Rightarrow u_{k+1} = \frac{1}{(k+2)!}; \quad u_k = \frac{1}{(k+1)!}$$

$$\rho = \lim_{k\to+\infty} \left(\frac{\frac{1}{(k+2)!}}{\frac{1}{(k+1)!}} \right) = \lim_{k\to+\infty} \left(\frac{(k+1)!}{(k+2)!} \right)$$

$$= \lim_{k\to+\infty} \left(\frac{1}{(k+2)} \right)$$

$$= 0 < 1 \Rightarrow \text{the series converges} \qquad (\because \rho < 1 \Rightarrow \text{the series converges})$$

Q 7: Find ρ by using the Limit Comparison Test where $\sum a_n = \frac{1}{2n-1}$ *and* $\sum b_n = \frac{1}{2n}$.

Solution:

$$\sum a_n = \frac{1}{2n-1} \quad and \quad \sum b_n = \frac{1}{2n}$$

$$\rho = \lim_{n\to+\infty} \frac{\frac{1}{2n-1}}{\frac{1}{2n}} = \lim_{n\to+\infty} \frac{2n}{2n-1} \qquad \left(\because \text{ The Limit Comparison test: } \rho = \lim_{n\to+\infty} \frac{a_n}{b_n}\right)$$

$$= \lim_{n\to+\infty} \frac{2}{2-\frac{1}{n}}$$

$$= \frac{2}{2-\frac{1}{\infty}} = \frac{2}{2} \qquad \left(\because \frac{a}{\infty} = 0\right)$$

$$= 1 \qquad \left(\therefore \rho \text{ is finite and } \rho > 0\right)$$

Lecture No. 44: Alternating Series; Conditional Convergence
Lecture No. 45: : Taylor and Maclaurin Series

Q 1: Find the radius of convergence for the following power series: $\sum_{1}^{\infty} \frac{n!\,.\,x^n}{(3n)!}$.

Solution:

Here $a_n = \frac{n!\,.\,x^n}{(3n)!}$,

so $a_{n+1} = \frac{(n+1)!\,.\,x^{n+1}}{(3n+3)!}$.

Now,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,.\,x^{n+1}}{(3n+3)!}\,.\,\frac{(3n)!}{n!\,.\,x^n}\right|,$$

$$= \lim_{n\to\infty}\frac{(n+1)n!}{(3n+3)(3n+2)(3n+1)(3n)!}\,.\,\frac{(3n)!}{n!}|x|\,,$$

$$= \lim_{n\to\infty}\frac{n+1}{(3n+3)(3n+2)(3n+1)}\,|x|,$$

$$= 0.$$

Thus the series converges absolutely $\forall\ x$ and radius of convergence $=\infty$.

Q 2: Show that $\sum_{1}^{\infty}|a_n|$ is divergent for the following alternating series: $\sum_{1}^{\infty}\frac{(-1)^n\ .\ n^n}{2n!}$.

Solution:

Here $a_n = \frac{(-1)^n\ .\ n^n}{2n!}$,

so $|a_n| = \frac{n^n}{2n!}$, and $|a_{n+1}| = \frac{(n+1)^{n+1}}{2(n+1)!}$.

Now $$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{(n+1)^{n+1}}{2(n+1)!}\,.\,\frac{2n!}{n^n}$$

$$= \lim_{n\to\infty}\ (\frac{n+1}{n})^n,$$

$$= \lim_{n\to\infty}\ (1+\frac{1}{n})^n,$$

$$= e > 1.$$

Thus the given series is diverges.

Q 3: Find the first two terms of Tylor series for $f(x) = \ln x$ at $x = 2$.

Solution:

$\because \quad f(x) = \ln x, \qquad f(2) = \ln 2,$

$\Rightarrow \; f'(x) = \dfrac{1}{x}, \qquad f'(2) = \dfrac{1}{2}.$

$\because$ Taylor polynomial for f about $x = a$:

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

$\Rightarrow p_0(x) = f(2) = \ln 2,$

$\Rightarrow p_1(x) = f(2) + f'(2)(x-2) = \ln 2 + \dfrac{1}{2}(x-2).$

Q 4: Find the first four terms of the Taylor series generated by f at $x = 2$ where

$$f(x) = \frac{1}{x+1}.$$

Solution:

$\because \quad f(x) = \dfrac{1}{x+1} = (x+1)^{-1} \qquad$ and $\qquad f(2) = \dfrac{1}{2+1} = \dfrac{1}{3},$

$\Rightarrow \quad f'(x) = -(x+1)^{-2} = -\dfrac{1}{(x+1)^2} \qquad$ and $\qquad f'(2) = -\dfrac{1}{(2+1)^2} = -\dfrac{1}{3^2},$

$\Rightarrow \quad f''(x) = 2(x+1)^{-3} = \dfrac{2}{(x+1)^3} \qquad$ and $\qquad f''(2) = \dfrac{2}{(2+1)^3} = \dfrac{2}{3^3},$

$\Rightarrow \quad f'''(x) = -6(x+1)^{-4} = -\dfrac{6}{(x+1)^4} \qquad$ and $\qquad f'''(2) = -\dfrac{6}{(2+1)^4} = -\dfrac{6}{3^4}.$

$\because$ Taylor Series

$$p_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$$

$$= \frac{1}{3} + \frac{-\frac{1}{3^2}}{1!}(x-2) + \frac{\frac{2}{3^3}}{2!}(x-2)^2 + \frac{-\frac{6}{3^4}}{3!}(x-2)^3 + \ldots$$

$$\Rightarrow \frac{1}{x+1} = \frac{1}{3} - \frac{1}{9}(x-2) + \frac{1}{27}(x-2)^2 - \frac{1}{81}(x-2)^3 + \ldots$$